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A General Galois Theory for Operations and Relations in Arbitrary Categories

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A General Galois Theory For Operations and Relations in Arbitrary Categories

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In this paper, we generalize the notions of polymorphisms and invariant relations to arbitrary categories. This leads us to a Galois connection that coincides with the classical case from universal algebra if the underlying category is the category of sets, but remains applicable no matter how the category is changed. In analogy to the situation in universal algebra, we characterize the Galois closed classes by local closures of clones of operations and local closures of what we will introduce as clones of (generalized) relations. Since the approach is built on purely category-theoretic properties, we will also discuss the dualization of our notions.

1 Introduction

First, it should be noted that the results of this paper are mainly taken from the authors Ph.D. thesis [Ker11], where most of this paper's content is presented in the context of a general duality theory for clones. Although we will refer to this duality theory as one of the motivations for this paper, the Galois theory itself will be presented independently from this context.

For a given set A, the notion of an operation to preserve a relation induces a Galois connection Pol-Inv that we can apply to operations over A and relations on A. The Galois closed classes are local closures of clones of operations and so-called clones of relations [Pös79, Pös80]. Nicknamed the "most basic Galois connection in algebra" [MMT87], there have been many attempts to generalize Pol-Inv or to transfer it to situations in which the operations are not functions over a set. For instance, in [PR00], the authors build a general Galois theory for cofunctions (i.e., functions from a set A to the union of finitely many disjoint copies of A) and what they define as corelations. Other examples are the investigation of a similar Galois theory for partial operations or multivalued functions [Ros83b, Röß00, Ros83a, Bör88, Cou05].

Here, we present an approach in which the notion of relations, that of preserving and the corresponding Galois connection are lifted to arbitrary categories. In fact, our theory will be applicable for operations over any given object in a category $\mathcal{C}$ as long as all finite non-empty powers of this object also exist in $\mathcal{C}$. We will show that our generalized Galois connection coincides with Pol-Inv if the category is the category of sets, and we demonstrate how the results from examples such as [PR00] follow directly from our theory.

For two reasons, the author of this paper claims that the generalization of Pol-Inv is useful even for those that are only interested in the usual scenario, i.e, the situation in the category of sets. On the one hand, it allows us to treat clones over sets abstractly (which has proven itself to be useful in many scenarios) while still having a tool analogue to Pol-Inv. On the other hand, each clone on a set A can be dualized to a so-called clone of dual operations, which, depending on the situation, can make some problems much easier to solve [Ker11]. However, the underlying category changes in the process of dualization, so we cannot apply any of the powerful techniques that Pol-Inv provides once the clone is dualized. In contrast, a general Galois theory based on purely category-theoretic properties can be dualized with the clone and is therefore still applicable.

The generalization of the Galois theory with all the corresponding notions will be done in Section 3. After we have succeeded in showing the desired results, we can apply the Duality Principle to obtain the dual results without any extra work. This will be done and discussed in Section 4. In this context, we will also point out how the duality enables us to solve some problems in an easier fashion.

On our way through Section 3 and 4, we will illustrate our steps with several examples.

2 Preliminaries

After recalling the rudimentary basics of clone theory in Subsection 2.1 and introducing our category-theoretic notation in Subsection 2.2, we will use Subsection 2.3 to raise the notion of a clone to categories, which will be the basis for our upcoming work.

2.1 Clones over Sets

Until the end of this subsection, let A be a non-empty set. For $n \in \mathbb{N}_+$, denote by $O_A^{(n)}$ the set of all n-ary operations over A and set $O_A := \bigcup_{n\in\mathbb{N}_+} O_A^{(n)}$. Note that O_A does not contain nullary operations.

The i-th argument of an n-ary operation f is said to be *non-essential* if

$$f(x_1, \ldots, x_n) \approx f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n).$$

An argument is called *essential* if it is not nonessential. Moreover, an operation is said to be *essentially k-ary* if it has exactly k essential arguments.

A subset $C \subseteq O_A$ is a called a *clone* (or *clone of operations*) if it contains all the projection mappings

$$\pi_i^n \colon A^n \to A : (x_1, \ldots, x_n) \mapsto x_i$$

and is closed with respect to superposition of operations in the following sense: For an n-ary operation $f \in C$ and k-ary operations $f_1, \ldots, f_n \in C$, the k-ary operation

$f(f_1,\dots,f_n)$, defined by setting

$$f(f_1,\dots,f_n)(x_1,\dots,x_k) := f(f_1(x_1,\dots,x_k),\dots,f_n(x_1,\dots,x_k)),$$

is also in C.

For each $F \subseteq O_A$, there is a least clone containing F. We denote this clone by $\mathrm{Clo}(F)$, and we say that F *generates* $\mathrm{Clo}(F)$. Note that $\mathrm{Clo}(F)$ can be interpreted as the set of term functions of $\langle A, F\rangle$. Hence, the clones on a set A represent all possible different behaviours of algebras with carrier set A.

It is easy to see that the clones over a set A form a lattice that we will denote by $\mathcal{L}_A$. On a two-element set, there are countably many clones, and the lattice was completely described by E. Post in [Pos41]. However, for $|A| \geq 3$, there are continuum many clones, and a full description of these lattices seems to be hopeless, even for $|A| = 3$. For more details on clone theory, we refer to [PK79] and [Sze86].

We will now see that there is a correspondence between clones of operations and certain sets of relations:

Denote by $R_A^{(n)}$ the set of all n-ary relations on A and set $R_A := \bigcup_{n\in\mathbb{N}_+} R_A^{(n)}$.

Definition 2.1. An operation $f \in O_A^{(n)}$ is said to *preserve* a relation $\sigma \in R_A^{(k)}$ if

$$\begin{pmatrix} \nu_{11} \\ \nu_{12} \\ \vdots \\ \nu_{1k} \end{pmatrix}, \dots, \begin{pmatrix} \nu_{n1} \\ \nu_{n2} \\ \vdots \\ \nu_{nk} \end{pmatrix} \in \sigma \Longrightarrow \begin{pmatrix} f(\nu_{11},\nu_{21},\dots,\nu_{n1}) \\ f(\nu_{12},\nu_{22},\dots,\nu_{n2}) \\ \vdots \\ f(\nu_{1k},\nu_{2k},\dots,\nu_{nk}) \end{pmatrix} \in \sigma.$$

For $F \subseteq O_A$ and $R \subseteq R_A$, define

$$\begin{aligned} \operatorname{Inv} F &:= \{\sigma \in R_A \mid \forall f \in F : f \rhd \sigma\}, \\ \operatorname{Pol} R &:= \{f \in O_A \mid \forall \sigma \in R : f \rhd \sigma\}. \end{aligned}$$

In terms of algebras, a k-ary relation σ belongs to $\operatorname{Pol} F$ if and only if σ forms a subalgebra of $\langle A, F\rangle^k$.

Obviously, Pol-Inv is a Galois connection between operations and relations. If A is a finite set, then the Galois closed classes are the clones of operations and the so-called clones of relations.

Definition 2.2. A relation $\sigma \in R_A^{(n)}$ is called a *diagonal relation on A* if there exists an equivalence relation $\theta \subseteq A \times A$ with

$$\sigma = \left\{ \left(\begin{matrix} a_1 \\ \vdots \\ a_n \end{matrix} \right) \in A^n \,\middle|\, \forall (i,j) \in \theta : a_i = a_j \right\}.$$

A *clone of relations on A* is a set of relations $R \subseteq R_A$ that contains all diagonal relations and is closed under direct (Cartesian) products, permutations and identifications of components.

Thus, for a finite set A, the clone lattice $\mathcal{L}_A$ is dually isomorphic to the lattice of clones of relations. In the case $|A| = \infty$, the Galois closed classes of Pol-Inv are local closures of the clones of operations and local closures of the clones of relations, and the lattices formed by these locally closed sets are the ones that are dually isomorphic. For more details on Pol-Inv, we refer to [Pös79] and [Pös80].

2.2 Category theory

We assume that the reader is familiar with the basic terminology of category theory. In this section, we only introduce our notation. For an object $\mathbf{A}$ in a category $\mathcal{C}$, we denote by $\mathbf{A}^n$ the n-th power of $\mathbf{A}$ (provided it exists) and by $\pi_i^n\colon \mathbf{A}^n \to \mathbf{A}$ ($i \in \{1,\dots,n\}$) the associated projection morphisms. For morphisms $f_1,\dots,f_n$ from the same object $\mathbf{B} \in \mathcal{C}$ to $\mathbf{A}$, we denote by $\langle f_1,\dots,f_n\rangle\colon \mathbf{B} \to \mathbf{A}^n$ the *tupling* of $f_1,\dots,f_n$.

Now, let $k_1,\dots,k_n \in \mathbb{N}_+$ and let $f_1,\dots,f_n$ be morphisms with $f_i\colon \mathbf{A}^{k_i} \to \mathbf{A}$ for all $i \in \{1,\dots,n\}$. Set $m := \sum_{i=1}^n k_i$ and $m_j := \sum_{i=1}^j k_i$ for each $j \in \{1,\dots,n-1\}$. We define the *expanded tupling* $\langle\!\langle f_1,\dots,f_n\rangle\!\rangle\colon \mathbf{A}^m \to \mathbf{A}^n$ by setting

$$\langle\!\langle f_1,\dots,f_n\rangle\!\rangle := \langle f_1 \circ \langle \pi_1^m,\dots,\pi_{m_1}^m\rangle,\dots,f_n \circ \langle \pi_{m_{n-1}+1}^m,\dots,\pi_m^m\rangle\rangle.$$

Note that, for $h_1,\dots,h_m\colon \mathbf{C} \to \mathbf{A}$, we have the following equation:

$$\langle\!\langle f_1,\dots,f_n\rangle\!\rangle \circ \langle h_1,\dots,h_m\rangle = \big\langle f_1 \circ \langle h_1,\dots,h_{m_1}\rangle,\dots,f_n \circ \langle h_{m_{n-1}+1},\dots,h_m\rangle\big\rangle\,.$$

Dually, for an object $\mathbf{X} \in \mathcal{C}$, we denote by $n \cdot \mathbf{X}$ the n-th copower of $\mathbf{X}$ (provided it exists) and by $\iota_i^n\colon \mathbf{X} \to n \cdot \mathbf{X}$ ($i \in \{1,\dots,n\}$) the associated injection morphisms. For morphisms $g_1,\dots,g_n$ from $\mathbf{X}$ to the same object $\mathbf{Y} \in \mathcal{C}$, we denote by $[g_1,\dots,g_n]\colon n \cdot \mathbf{X} \to \mathbf{Y}$ the *cotupling of* $g_1,\dots,g_n$.

For two objects $\mathbf{A},\mathbf{B} \in \mathcal{C}$, we write $\mathbf{A} \lessdot \mathbf{B}$ if there exists a monomorphism from $\mathbf{A}$ to $\mathbf{B}$ and we write $\mathbf{A} \leqslant \mathbf{B}$ if there exists an epimorphism from $\mathbf{B}$ to $\mathbf{A}$.

By $\mathcal{C}^{op}$, we denote the *opposite category of* $\mathcal{C}$ (i.e., the category obtained from $\mathcal{C}$ by reversing all morphisms).

The category of sets will be denoted by $\mathcal{S}et$.

2.3 Clones in categories

In this section, we will generalize the notion of operations and clones to categories.

Definition 2.3. Let $n \in \mathbb{N}_+$. A morphism $f\colon \mathbf{A}^n \to \mathbf{A}$ is called an *n-ary operation over* $\mathbf{A}$. Denote by $O_{\mathbf{A}}^{(n)}$ the set of all n-ary operations over $\mathbf{A}$, define $O_{\mathbf{A}} := \bigcup_{n\in\mathbb{N}_+} O_{\mathbf{A}}^{(n)}$ and, for $F \subseteq O_{\mathbf{A}}$, set $F^{(n)} := F \cap O_{\mathbf{A}}^{(n)}$.

We can also extend the notion of essential and nonessential arguments:

Definition 2.4. Let $\widetilde{\pi}_i^n\colon \mathbf{A}^n \to \mathbf{A}^{n-1}$ denote the morphism defined by setting

$$\widetilde{\pi}_i^n := \langle \pi_1^n,\dots,\pi_{i-1}^n,\pi_{i+1}^n,\dots,\pi_n^n\rangle.$$

The i-th argument of an n-ary operation f is said to be *nonessential* if there exists an $(n-1)$-ary operation $f' \in O_{\mathbf{A}}$ such that $f = f' \circ \widetilde{\pi}_i^n$. An argument is called *essential* if it is not nonessential. Moreover, we say that an operation is *essentially n-ary* if it has exactly n essential arguments.

For $\mathcal{C} = \mathcal{S}et$, this definition coincides with the usual definition of nonessential and essential arguments.

Definition 2.5. A set C of operations over $\mathbf{A}$ is called a *clone of operations*, written $C \leq O_{\mathbf{A}}$, if C contains all the projection morphisms $\pi_i^n \colon \mathbf{A}^n \to \mathbf{A}$ and, for $f \in C^{(n)}$, $f_1, \ldots, f_n \in C^{(k)}$, the *superposition* $f \circ \langle f_1, \ldots, f_n \rangle$ is also in C.

If $\mathcal{C}$ is the category of sets, then this definition coincides with the usual notion of a clone. It is easy to verify that the clones over an object $\mathbf{A}$ form a complete lattice with respect to inclusion. The top element of the lattice is the *full clone* $O_{\mathbf{A}}$ and the bottom element is the clone that contains only the projection morphisms.

Definition 2.6. Denote by $L_{\mathbf{A}}$ the set of clones of operations over $\mathbf{A}$. Then, the ordered set $\mathcal{L}_{\mathbf{A}} := \langle L_{\mathbf{A}}, \subseteq \rangle$ is called the *lattice of clones over* $\mathbf{A}$.

Since clones are closed under arbitrary intersection, we can define the closure operator Clo that assigns to each subset $F \subseteq O_{\mathbf{A}}$ the least clone of operations over $\mathbf{A}$ that contains F. It is called the clone *generated by* F. For a single operation f, we write $\mathrm{Clo}(f)$ to mean $\mathrm{Clo}(\{f\})$.

Examples 2.7.

(i) If $\mathcal{C} = \mathcal{S}et$, then $O_{\mathbf{A}}$ is the full clone on the set A and $\mathcal{L}_{\mathbf{A}}$ is the usual clone lattice.
(ii) If $\mathcal{C}$ is a quasivariety of algebras, then $O_{\mathbf{A}}$ is the so-called centralizer clone of the algebra $\mathbf{A}$ and $\mathcal{L}_{\mathbf{A}}$ is the lattice of subclones of $O_{\mathbf{A}}$.
(iii) If $\mathcal{C}$ is a quasivariety of relational structures, then we have $O_{\mathbf{A}} = \mathrm{Pol}\, R$ and $\mathcal{L}_{\mathbf{A}}$ is the sublattice $\langle \mathrm{Pol}\, R] \leq \mathcal{L}_A$.
(iv) If $\mathcal{C}$ is a the category of topological spaces and $\mathbf{A} \in \mathcal{C}$, then $O_{\mathbf{A}}$ is the clone of the topological space $\mathbf{A}$ as investigated by W. Taylor in [Tay86].

The following proposition will be needed in the remainder of this paper.

Proposition 2.8. *Clones of operations are closed under expanded superposition. That is, for a clone $C \leq O_{\mathbf{A}}$, we have $f \circ \langle\!\langle f_1, \ldots, f_n \rangle\!\rangle \in C$ for all $f \in C^{(n)}$ and $f_1, \ldots, f_n \in C$.*

Proof. Let $k_1, \ldots, k_n \in \mathbb{N}_+$ such that $f_i \in O_{\mathbf{A}}^{(k_i)}$ for all $i \in \{1, \ldots, n\}$. Moreover, for $j \in \{1, \ldots, n-1\}$, set $m_j := \sum_{i=1}^{j} k_i$ and $m := \sum_{i=1}^{n} k_i$. But now, we have

$$\begin{aligned} f \circ \langle\!\langle f_1, \ldots, f_n \rangle\!\rangle &= f \circ \langle\!\langle f_1, \ldots, f_n \rangle\!\rangle \circ \langle \pi_1^m, \pi_2^m, , \ldots, \pi_m^m \rangle \\ &= f \circ \langle \underbrace{f_1 \circ \langle \pi_1^m, \ldots, \pi_{m_1}^m \rangle}_{\in C}, \ldots, \underbrace{f_n \circ \langle \pi_{m_{n-1}+1}^m, \ldots, \pi_m^m \rangle}_{\in C} \rangle \in C. \end{aligned}$$

□

Having written everything in purely category-theoretic terms, we can also dualize all the notions.

Definition 2.9. Let $n \in \mathbb{N}_+$. An *n-ary dual operation over* $\mathbf{X}$ (or *cooperation over* $\mathbf{X}$) is a morphism from $\mathbf{X}$ to $n \cdot \mathbf{X}$. Denote by $\overline{O}_{\mathbf{X}}^{(n)}$ the set of all n-ary dual operations over $\mathbf{X}$, define $\overline{O}_{\mathbf{X}} := \bigcup_{n \in \mathbb{N}_+} \overline{O}_{\mathbf{X}}^{(n)}$ and, for $G \subseteq \overline{O}_{\mathbf{X}}$, set $G^{(n)} := G \cap \overline{O}_{\mathbf{X}}^{(n)}$.

For a dual operation, we can also speak of essential and nonessential arguments:

Definition 2.10. Let $\widetilde{\iota}_i^n\colon (n-1)\cdot\mathbf{X}\to n\cdot\mathbf{X}$ denote the operation defined by setting

$$\widetilde{\iota}_i^n := [\iota_1^n,\ldots,\iota_{i-1}^n,\iota_{i+1}^n,\ldots,\iota_n^n].$$

The i-th argument of an n-ary dual operation g is said to be *nonessential* if there exists an $(n-1)$-ary dual operation $g'\in\overline{O}_{\mathbf{X}}$ such that $g=\widetilde{\iota}_i^n\circ g'$. An argument is called *essential* if it is not nonessential. Moreover, we say that a dual operation is *essentially n-ary* if it has exactly n essential arguments.

Definition 2.11. A set C of dual operations over $\mathbf{X}$ is a *clone of dual operations (or coclone)* if it contains all the injection morphisms and, for $g\in C^{(n)}$ and $g_1,\ldots,g_n\in C^{(k)}$, the *superposition* $[g_1,\ldots,g_n]\circ g$ is also in C.

Examples 2.12.

(i) If $\mathbf{X}$ is a set in the category of sets, then a clone of dual operations over $\mathbf{X}$ is a coclone as introduced in [Csá85].
(ii) If C is a clone of operation over $\mathbf{A}$ in $\mathcal{C}$, then C^{op} is a clone of dual operations over $\mathbf{A}$ in $\mathcal{C}^{op}$.

Analogue to the closure operator Clo on the clones of operations, we can define $\overline{\mathrm{Clo}}$: For a set of dual operations $G\subseteq\overline{O}_{\mathbf{X}}$, we denote by $\overline{\mathrm{Clo}}(G)$ the least clone of dual operations that contains G. Again, for a single dual operation, we write $\overline{\mathrm{Clo}}(g)$ instead of $\overline{\mathrm{Clo}}(\{g\})$.

Definition 2.13. Denote by $\overline{L}_{\mathbf{X}}$ the set of clones of dual operations over $\mathbf{X}$. The ordered set $\overline{\mathcal{L}}_{\mathbf{X}}:=\langle L_{\mathbf{X}},\subseteq\rangle$ is called the *lattice of clones of dual operations over* $\mathbf{X}$.

In [Ker11], it is discussed how clones over sets can be efficiently dualized to clones of dual operations, and it is shown that this technique can be used to solve clone-theoretic problems. In this context, a general Galois theory for operations and relations is introduced and incorporated into the approach. In the next sections, we present this general Galois theory independently from the context of clone dualities.

3 A General Galois Theory for Operations and Relations in Categories

For the whole section, let $\mathcal{C}$ be a category with an object $\mathbf{A}$ such that all finite non-empty powers of $\mathbf{A}$ are also in $\mathcal{C}$.

3.1 Generalized Relations

To understand the idea of our approach, let us note that one can interpret relations in the usual sense as sets of mappings. If we do so, we can say that σ is a k-ary relation on the set A if σ is a subset of $A^{\{1,\ldots,k\}}$. Thus, a relation on A is nothing else but a set of morphisms from the object $\{1,\ldots,k\}$ to the object A in the category of sets, i.e., it is a subset of $\mathcal{Set}(\{1,\ldots,k\},A)$. This is precisely the view on relations that we will now use to generalize relations on sets to relations on the object $\mathbf{A}$: Analogue to defining k-ary relations to be sets of mappings from the set $\{1,\ldots,k\}$ to the set A, we will define a relation of type $\mathbf{B}\in\mathcal{C}$ to be a set of morphisms from the object $\mathbf{B}$ to the object $\mathbf{A}$:

Definition 3.1. Let $\mathbf{B}\in\mathcal{C}$. A *relation of type* $\mathbf{B}$ *on* $\mathbf{A}$ is a subset of $\mathcal{C}(\mathbf{B},\mathbf{A})$. Denote the class of all relations of type $\mathbf{B}$ on $\mathbf{A}$ by $\mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$.

We will now define the notion of invariant relations on $\mathbf{A}$ by generalizing the usual notion of invariant relations. Recall that an n-ary function f on a set A is said to preserve a k-ary relation σ if

$$\begin{pmatrix}\nu_{11}\\ \nu_{12}\\ \vdots\\ \nu_{1k}\end{pmatrix},\ldots,\begin{pmatrix}\nu_{n1}\\ \nu_{n2}\\ \vdots\\ \nu_{nk}\end{pmatrix}\in\sigma\Longrightarrow\begin{pmatrix}f(\nu_{11},\nu_{21},\ldots,\nu_{n1})\\ f(\nu_{12},\nu_{22},\ldots,\nu_{n2})\\ \vdots\\ f(\nu_{1k},\nu_{2k},\ldots,\nu_{nk})\end{pmatrix}\in\sigma.$$

If we interpret the relation σ as a subset of $\mathcal{Set}(\{1,\ldots,k\},A)$, then we can express the condition of preserving by using the tupling:

$$f\rhd\sigma\Longleftrightarrow f\circ\langle r_1,\ldots,r_n\rangle\in\sigma\text{ for all }r_1,\ldots,r_n\in\sigma.$$

Since this notion of preserving relies on purely category-theoretic properties, we can lift it to other categories.

Definition 3.2. Let σ be a relation of type $\mathbf{B}$ on $\mathbf{A}$ and let $f\in O_{\mathbf{A}}^{(n)}$. Say that σ is *invariant for* f or that f *preserves* σ, written $f\rhd\sigma$, if

$$f\circ\langle r_1,\ldots,r_n\rangle\in\sigma$$

whenever $r_1,\ldots,r_n\in\sigma$. Furthermore, a set of operations $F\subseteq O_{\mathbf{A}}$ is said to preserve σ, written $F\rhd\sigma$, if every $f\in F$ preserves σ.

Clearly, for $\mathcal{C}$ being the category of sets and $\mathbf{B}=\{1,\ldots,k\}$, this notion coincides with the usual notion of f preserving a k-ary relation.

Note that the projection morphisms preserve any relation on $\mathbf{A}$.

Definition 3.3. For $F\subseteq O_{\mathbf{A}}$ and $\sigma\in\mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$, define

$$\Gamma_F(\sigma):=\bigcap\{\sigma'\in\mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}\mid\sigma\subseteq\sigma',F\rhd\sigma'\}.$$

It is easy to see that the intersection of relations preserved by some $F\subseteq O_{\mathbf{A}}$ is again preserved by F. Furthermore, the *full relation* $\mathcal{C}(\mathbf{B},\mathbf{A})$ is invariant for each set of operations over $\mathbf{A}$. Thus, for each $F\subseteq O_{\mathbf{A}}$ and each relation $\sigma\in\mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$, $\Gamma_F(\sigma)$ is the least relation on $\mathbf{A}$ of type $\mathbf{B}$ that is preserved by F and contains σ.

We will now show that the superposition of operations preserves σ if each operation in the superposition preserves σ.

Proposition 3.4. *Let $f_1, \ldots, f_n \in O_{\mathbf{A}}^{(k)}$, $f \in O_{\mathbf{A}}^{(n)}$ and let $\sigma \in \mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$. Then*

$$f, f_1, \ldots, f_n \rhd \sigma \Longrightarrow f \circ \langle f_1, \ldots, f_n \rangle \rhd \sigma.$$

Proof. Let $f, f_1, \ldots, f_n$ preserve σ. For $r_1, \ldots, r_n \in \sigma$, we have

$$f \circ \langle f_1, \ldots, f_n \rangle \circ \langle r_1, \ldots, r_k \rangle = f \circ \langle \underbrace{f_1 \circ \langle r_1, \ldots, r_k \rangle}_{\in \sigma}, \ldots, \underbrace{f_n \circ \langle r_1, \ldots, r_k \rangle}_{\in \sigma} \rangle \in \sigma.$$

Hence, $f \circ \langle f_1, \ldots, f_n \rangle$ preserves σ. □

The following corollary is an almost trivial consequence, but it is important as it provides us with a very efficient technique to show that a given operation cannot generate another given operation:

Corollary 3.5. *Let $f, f' \in O_{\mathbf{A}}$ and let $\sigma \in \mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$. If $f \rhd \sigma$ but $f' \not\rhd \sigma$, then $f' \notin \mathrm{Clo}(f)$.*

Proof. Assume $f' \in \mathrm{Clo}(f)$, that is, f' is a superposition of f and the projection morphisms. Since f and the projection morphisms preserve σ, it follows by Proposition 3.4 that we also have $f' \rhd \sigma$, a contradiction. □

Now, we want to define clones of relations on $\mathbf{A}$ analogue to the situation in the category of sets. In [Pös79], it was observed that a clone of relations in the usual sense can be expressed as follows if we take the point of view we described above, namely to think of k-ary relations as sets of mappings from $\{1, \ldots, k\}$ to A:

Proposition 3.6 ([Pös79])**.** *Let R be a set of (finitary) relations on a set A where each $\sigma \in R^{(k)}$ is interpreted as a set of mappings from $\{1, \ldots, k\}$ to A. Then, R is a clone of relations on A if and only if*

(i) $\emptyset \in R$,

(ii) *R is closed under general superposition, that is, the following holds: Let I be an index set, let $\sigma_i \in R^{(k_i)}$ $(i \in I)$ and let $\varphi\colon \{1, \ldots, k\} \to \alpha$ and $\varphi_i\colon \{1, \ldots, k_i\} \to \alpha$ be mappings where α is some cardinal number. Then, the relation $\bigwedge^{\varphi}_{(\varphi_i)_{i \in I}} (\sigma_i)_{i \in I}$ defined by*

$$\bigwedge\nolimits^{\varphi}_{(\varphi_i)_{i \in I}} (\sigma_i)_{i \in I} := \{ r \circ \varphi \mid \forall\, i \in I : r \circ \varphi_i \in \sigma_i,\, r \in A^{\alpha} \}$$

belongs to R.

To transfer this definition to our general environment, we introduce the notion of a typeclass.

Definition 3.7. A *typeclass* is a non-empty subclass $\mathbb{T} \subseteq \mathcal{C}$ in which any two different objects are non-isomorphic.

In other words, a typeclass is a non-empty subclass of a skeleton.

Examples 3.8.

(i) Each skeleton is a typeclass.
(ii) If $\mathcal{C} = \mathcal{Set}$, then $\mathbb{T} := \{\{1, \ldots, k\} \mid k \in \mathbb{N}_+\}$ is a typeclass.
(iii) If $(\mathcal{C}, U)$ is a concrete category, then a representation system of

$$\{\mathbf{A} \in \mathcal{C} \mid |U(\mathbf{A})| < \infty\}/\cong$$

is a typeclass.
(iv) If $\mathcal{C}$ is the category of finite distributive lattices, then

$$\mathbb{T} := \{\langle \mathfrak{P}(\{1, \ldots, k\}), \cup, \cap\rangle \mid k \in \mathbb{N}_+\}$$

is a typeclass. Note that $\mathbb{T}$ is, up to isomorphism, the class of all finite Boolean lattices.

Definition 3.9. For a typeclass $\mathbb{T}$,

$$\mathrm{R}_{\mathbf{A}}^{\mathbb{T}} := \bigcup_{\mathbf{B} \in \mathbb{T}} \mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$$

is called *the class of relations of the typeclass* $\mathbb{T}$ *on* $\mathbf{A}$.

For a class of relations $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ and $\mathbf{B} \in \mathbb{T}$, we write $R^{(\mathbf{B})}$ to indicate $R \cap \mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$.

Note that we have $\emptyset \in \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ since $\emptyset$ is a relation of type $\mathbf{B}$ for all $\mathbf{B} \in \mathbb{T}$ and $\mathbb{T}$ is non-empty by definition.

Examples 3.10. Let $\mathcal{C}$ be the category of sets.

(i) If we choose $\mathbb{T} := \{\{1, \ldots, k\} \mid k \in \mathbb{N}_+\}$, then $\mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ coincides with the set of finitary relations in the usual sense.
(ii) If we choose $\mathbb{T}$ to be the class of all cardinal numbers (written as sets), then $\mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ coincides with the set of (possibly infinitary) relations in the usual sense.

We are now ready to define the notion of a clone of relations on $\mathbf{A}$ by generalizing Proposition 3.6 in a straight-forward way.

Definition 3.11. A class $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ is called a *clone of relations of the typeclass* $\mathbb{T}$ *on* $\mathbf{A}$, written $R \leq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, if

(i) $\emptyset \in R$,
(ii) R is closed under *general superposition*, that is, the following holds: Let I be an index class, let $\sigma_i \in R^{(\mathbf{B}_i)}$ $(i \in I)$ and let $\varphi\colon \mathbf{B} \to \mathbf{C}$ and $\varphi_i\colon \mathbf{B}_i \to \mathbf{C}$ be morphisms where $\mathbf{C} \in \mathcal{C}$ and $\mathbf{B} \in \mathbb{T}$. Then, the relation $\bigwedge^{\varphi}_{(\varphi_i)_{i\in I}}(\sigma_i)_{i \in I} \in \mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$ defined by

$$\bigwedge\nolimits^{\varphi}_{(\varphi_i)_{i\in I}}(\sigma_i)_{i\in I} := \bigwedge\nolimits^{\varphi}_{(\varphi_i)}(\sigma_i) := \{r \circ \varphi \mid \forall\, i \in I : r \circ \varphi_i \in \sigma_i,\, r \in \mathcal{C}(\mathbf{C}, \mathbf{A})\}$$

belongs to R.

Note that it suffices to check the second condition for all $\mathbf{C}$ in some skeleton of $\mathcal{C}$. In fact, if such objects exist, it is enough to consider those $\mathbf{C}$ in a skeleton that are maximal with respect to $\leqslant$ (recall that we write $\mathbf{C}_1 \leqslant \mathbf{C}_2$ if there exists an epimorphism from $\mathbf{C}_2$ to $\mathbf{C}_1$).

Example 3.12. Let $\mathcal{C}$ be the category of sets and let $\mathbf{A} \in \mathcal{C}$. If we choose $\mathbb{T}$ as in case (i) of Example 3.10, then our notion of a clone of relations coincides with the usual notion of a clone of finitary relations. If we choose $\mathbb{T}$ as in case (ii), then our notion coincides with the usual notion of a clone of (possibly infinitary) relations [Ros72].

For a given typeclass $\mathbb{T}$, it is obvious that $\mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ is a clone of relations. Furthermore, it is easy to see that the intersection of clones of relations is again a clone of relations. Thus, for $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, there exists a clone of relations that is the least clone among those that contain R.

Definition 3.13. Denote by $\mathrm{Clo}^{\mathbb{T}} \colon \mathfrak{P}(\mathrm{R}_{\mathbf{A}}^{\mathbb{T}}) \to \mathfrak{P}(\mathrm{R}_{\mathbf{A}}^{\mathbb{T}})$ the operator that maps each $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ to the least clone of relations that contains R. Say that $\mathrm{Clo}^{\mathbb{T}}(R)$ is the clone of relations *generated by* R.

Hence, for a given typeclass $\mathbb{T}$, the clones of relations on $\mathbf{A}$ form a complete lattice with respect to inclusion.

Definition 3.14. Denote by $L_{\mathbf{A}}^{*\mathbb{T}}$ the class of clones of relations of the typeclass $\mathbb{T}$ on $\mathbf{A}$. Then, $\mathcal{L}_{\mathbf{A}}^{*\mathbb{T}} := \langle L_{\mathbf{A}}^{*\mathbb{T}}, \subseteq \rangle$ is called the *lattice of clones of relations of the typeclass* $\mathbb{T}$ *on* $\mathbf{A}$.

Clearly, $\mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ is the greatest clone of relations on $\mathbf{A}$, whereas the least clone of relations on $\mathbf{A}$ is $\mathrm{Clo}^{\mathbb{T}}(\emptyset)$. The latter contains precisely the empty relation and all relations that arise from the general superposition of relations with an empty index class. That is,

$$\mathrm{Clo}^{\mathbb{T}}(\emptyset) = \{\emptyset\} \cup \{r \circ \varphi \mid r \in \mathcal{C}(\mathbf{C}, \mathbf{A}), \varphi \colon \mathbf{B} \to \mathbf{C}, \mathbf{B} \in \mathbb{T}, \mathbf{C} \in \mathcal{C}\}.$$

In the scenarios from 3.12 (i.e., the universal algebra case with finitary or infinitary relations), these are precisely the diagonal relations.

3.1.1 The Generalized Galois Connection $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$

Until the end of this section, let $\mathbb{T}$ be a typeclass of $\mathcal{C}$.

Definition 3.15. We define the two operators $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \colon \mathfrak{P}(O_{\mathbf{A}}) \to \mathfrak{P}(\mathrm{R}_{\mathbf{A}}^{\mathbb{T}})$ and $\mathrm{Pol}_{\mathbf{A}} \colon \mathfrak{P}(\mathrm{R}_{\mathbf{A}}^{\mathbb{T}}) \to \mathfrak{P}(O_{\mathbf{A}})$ as follows: For $F \subseteq O_{\mathbf{A}}$ and $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, set

$$\begin{aligned}
\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F &:= \{\sigma \in \mathrm{R}_{\mathbf{A}}^{\mathbb{T}} \mid \forall f \in F : f \rhd \sigma\}, \\
\mathrm{Pol}_{\mathbf{A}} R &:= \{f \in O_{\mathbf{A}} \mid \forall \sigma \in R : f \rhd \sigma\}.
\end{aligned}$$

For $\mathbf{B} \in \mathcal{C}$ and $n \in \mathbb{N}_+$, we use the following notations:

$$\begin{aligned}
\mathrm{Inv}_{\mathbf{A}}^{(\mathbf{B})} F &:= \{\sigma \in \mathrm{R}_{\mathbf{A}}^{(\mathbf{B})} \mid \forall f \in F : f \rhd \sigma\}, \\
\mathrm{Pol}_{\mathbf{A}}^{(n)} R &:= \mathrm{Pol}_{\mathbf{A}} R \cap O_{\mathbf{A}}^{(n)}.
\end{aligned}$$

Note that $\mathrm{Pol}_{\mathbf{A}} R$ and $\mathrm{Inv}_{\mathbf{A}}^{(\mathbf{B})} F$ are always sets, while $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F$ can be a proper class, so the operators $\mathrm{Pol}_{\mathbf{A}}$ and $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$ constitute a Galois connection between the subsets of $O_{\mathbf{A}}$ and the subclasses of $\mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$.

For $\mathcal{C} = \mathcal{Set}$, it follows directly from the observations in Example 3.12 that $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$ coincides with Pol-Inv if we choose $\mathbb{T} = \{\{1, \ldots, k\} \mid k \in \mathbb{N}_+\}$ and that it coincides with the Galois connection from [Ros72] if we choose $\mathbb{T}$ to be the class of all positive cardinal numbers.

Examples 3.16.

(i) Let $\sigma := \{id_{\mathbf{A}}\}$. Then, $\mathrm{Pol}_{\mathbf{A}}\{\sigma\}$ is the set of all idempotent operations over $\mathbf{A}$. That is, $\mathrm{Pol}_{\mathbf{A}}\{\sigma\} = \{f \in O_{\mathbf{A}} \mid f \circ \langle id_{\mathbf{A}}, \ldots, id_{\mathbf{A}}\rangle = id_{\mathbf{A}}\}$.

(ii) Let $C \leq O_{\mathbf{A}}$. Note that $C^{(n)}$ is a relation of type $\mathbf{A}^n$. Now, $\mathrm{Pol}_{\mathbf{A}} C^{(n)}$ is the largest clone C' that agrees with C on its n-ary part, i.e., the largest clone C' with $C'^{(n)} = C^{(n)}$.

(iii) If two operations f, f' are essentially the same (i.e. after eliminating all non-essential arguments, the two operations arise from each other by a suitable permutation of arguments), then $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}\{f\} = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}\{f'\}$.

(iv) Let $\mathcal{C}$ be the category of finite distributive lattices and let $\mathbf{B} \in \mathcal{C}$. Let $\sigma \in \mathrm{R}_{\mathbf{A}}^{(\mathbf{B})}$ contain all morphisms $r \colon \mathbf{B} \to \mathbf{A}$ with $r(0^{\mathbf{B}}) = 0^{\mathbf{A}}$ and $r(1^{\mathbf{B}}) = 1^{\mathbf{A}}$. Then, $\mathrm{Pol}_{\mathbf{A}}\{\sigma\}$ is the set of all 01-homomorphisms over $\mathbf{A}$ (i.e, all operations over $\mathbf{A}$ that preserve the bottom and the top of the lattice).

Note that, in these examples, the set of polymorphisms always turned out to be a clone. This is something we know for the usual Pol-Inv, and we will shortly see that it is also true for $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$. Indeed, we will see that we can generalize almost every definition, lemma, proposition and theorem that holds for Pol-Inv.

Proposition 3.17. *Let $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, $F \subseteq O_{\mathbf{A}}$, $\mathbf{B}, \mathbf{C} \in \mathbb{T}$ and $s_1, s_2 \in \mathbb{N}_+$. For $s_1 \leq s_2$ and $\mathbf{B} \leqslant \mathbf{C}$, we have*

(i) $\mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{C})} F \subseteq \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{B})} F$,

(ii) $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}}^{(s_2)} R \subseteq \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}}^{(s_1)} R$.

Proof. (i) Let $h \in \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{C})} F$ be n-ary and let $\sigma \in \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{B})} R$. We need to show that h preserves σ. Since $\mathbf{B} \leqslant \mathbf{C}$, there exists an epimorphism $e \colon \mathbf{C} \to \mathbf{B}$. Let

$$\sigma' := \{r \circ e \mid r \in \sigma\}.$$

Note that σ' is a relation of type $\mathbf{C}$. First, we will show that σ' is preserved by F. Let $f \in F^{(m)}$ and let $r_1', \ldots, r_m' \in \sigma'$. Then, there exist $r_1, \ldots, r_m \in \sigma$ such that $r_j' = r_j \circ e$ for all $j \in \{1, \ldots, m\}$. But now,

$$f \circ \langle r_1', \ldots, r_m'\rangle = f \circ \langle r_1 \circ e, \ldots, r_m \circ e\rangle = \underbrace{f \circ \langle r_1, \ldots, r_m\rangle}_{\in \sigma} \circ e \in \sigma'.$$

Hence, $\sigma' \in \operatorname{Inv}_{\mathbf{A}}^{(\mathbf{C})} F$. Since $h \in \operatorname{Pol}_{\mathbf{A}} \operatorname{Inv}_{\mathbf{A}}^{(\mathbf{C})} F$, this means $h \triangleright \sigma'$. Finally, let $r_1, \ldots, r_n \in \sigma$. We have

$$h \circ \langle r_1, \ldots, r_n \rangle \circ e = h \circ \langle \underbrace{r_1 \circ e}_{\in \sigma'}, \ldots, \underbrace{r_n \circ e}_{\in \sigma'} \rangle \in \sigma'.$$

But now $h \circ \langle r_1, \ldots, r_n \rangle \circ e \in \sigma'$ implies that there exists $r \in \sigma$ such that

$$h \circ \langle r_1, \ldots, r_n \rangle \circ e = r \circ e.$$

Since e is an epimorphism, this implies $h \circ \langle r_1, \ldots, r_n \rangle = r \in \sigma$, and we are done.

(ii) For $f \in \operatorname{Pol}_{\mathbf{A}}^{(s_1)} R$, we have $f' := f \circ \langle \pi_1^{s_2}, \ldots, \pi_{s_1}^{s_2} \rangle \in \operatorname{Pol}_{\mathbf{A}}^{(s_2)} R$. The claim now follows from the observation that a relation is preserved by f if and only if it is preserved by f'. □

Definition 3.18. Let $F \subseteq O_{\mathbf{A}}$, $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, $s \geq 1$ and let $\mathbf{C} \in \mathcal{C}$. We define the following local closure operators:

$$\begin{aligned} \mathbf{C}\text{-Loc}\, F :=& \{f \in O_{\mathbf{A}}^{(n)} \mid n \geq 1, \forall\, r_1, \ldots, r_n \in \mathcal{C}(\mathbf{C}, \mathbf{A}) : \\ & \qquad \exists\, f' \in F : f \circ \langle r_1, \ldots, r_n \rangle = f' \circ \langle r_1, \ldots, r_n \rangle \}, \\ \text{s-LOC}^{\mathbb{T}} R :=& \{\sigma \in \mathrm{R}_{\mathbf{A}}^{\mathbb{T}} \mid \forall B \subseteq \sigma, |B| \leq s : \exists\, \sigma' \in R : B \subseteq \sigma' \subseteq \sigma\}. \end{aligned}$$

Furthermore, let

$$\operatorname{Loc}^{\mathbb{T}} F := \bigcap_{\mathbf{C} \in \mathbb{T}} \mathbf{C}\text{-Loc}\, F$$

and

$$\operatorname{LOC}^{\mathbb{T}} R := \bigcap_{s \in \mathbb{N}_+} \text{s-LOC}^{\mathbb{T}} R.$$

In other words, $\mathbf{C}$-Loc F is the set of all operations $f \in O_{\mathbf{A}}$ such that, for all tuplings $\langle r_1, \ldots, r_n \rangle$ of morphism from $\mathbf{C}$ to $\mathbf{A}$, there exists an operation $f' \in F$ such that f and f' cannot be distinguished if they are applied after $\langle r_1, \ldots, r_n \rangle$. Moreover, s-LOC$^{\mathbb{T}}$ R is the class of all relations $\sigma \in \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ such that, for every $B \subseteq \sigma$ with at most s elements, there exists a member σ' of R that agrees with σ on B and is contained in σ.

We will see later that $\operatorname{Loc}^{\mathbb{T}} C$ is a clone of operations whenever C is a clone of operations (Theorem 3.31, page 17). Similarly, we will see that $\operatorname{LOC}^{\mathbb{T}} R$ is a clone of relations whenever R is a clone of relations (Theorem 3.33, page 19).

Proposition 3.19. *If* $\mathbf{C}_1 \leqslant \mathbf{C}_2$*, then* $\mathbf{C}_2$-Loc $F \subseteq \mathbf{C}_1$-Loc F *for all* $F \subseteq O_{\mathbf{A}}$.

Proof. Let $f \in \mathbf{C}_2$-Loc F be n-ary and let $r_1, \ldots, r_n \in \mathcal{C}(\mathbf{C}_1, \mathbf{A})$. We need to show that there exists $f' \in F$ such that $f \circ \langle r_1, \ldots, r_n \rangle = f' \circ \langle r_1, \ldots, r_n \rangle$. Since $\mathbf{C}_1 \leqslant \mathbf{C}_2$, there exists an epimorphism $e \colon \mathbf{C}_2 \to \mathbf{C}_1$. For $i \in \{1, \ldots, n\}$, let $r_i' := r_i \circ e$. Since $r_i' \in \mathcal{C}(\mathbf{C}_2, \mathbf{A})$ and $f \in \mathbf{C}_2$-Loc F, there exists $f' \in F$ such that

$$f \circ \langle r_1', \ldots, r_n' \rangle = f' \circ \langle r_1', \ldots, r_n' \rangle.$$

Hence,

$$\begin{aligned} f \circ \langle r_1, \dots, r_n \rangle \circ e &= f \circ \langle r_1 \circ e, \dots, r_n \circ e \rangle \\ &= f \circ \langle r'_1, \dots, r'_n \rangle \\ &= f' \circ \langle r'_1, \dots, r'_n \rangle \\ &= f' \circ \langle r_1, \dots, r_n \rangle \circ e. \end{aligned}$$

But now, since e is an epimorphism, this implies $f \circ \langle r_1, \dots, r_n \rangle = f' \circ \langle r_1, \dots, r_n \rangle$. Thus, $f \in \mathbf{C}_1\text{-Loc}\, F$. $\square$

Lemma 3.20. *Let $n \in \mathbb{N}_+$ and let $F \subseteq O_{\mathbf{A}}$. If we have $\mathbf{A}^n \leqslant \mathbf{B}$ for some $\mathbf{B} \in \mathbb{T}$, then $(\mathrm{Loc}^{\mathbb{T}} F)^{(n)} = F^{(n)}$ (that is, $\mathrm{Loc}^{\mathbb{T}} F$ and F agree on their n-ary part).*

Proof. $F^{(n)} \subseteq (\mathrm{Loc}^{\mathbb{T}} F)^{(n)}$ is obvious. To show $(\mathrm{Loc}^{\mathbb{T}} F)^{(n)} \subseteq F^{(n)}$, let $f \in O_{\mathbf{A}}^{(n)}$ belong to $\mathrm{Loc}^{\mathbb{T}} F$. By assumption, there exists $\mathbf{B} \in \mathbb{T}$ with $\mathbf{A}^n \leqslant \mathbf{B}$. Let $e \colon \mathbf{B} \to \mathbf{A}^n$ be an epimorphism. Since $f \in \mathrm{Loc}^{\mathbb{T}} F \subseteq \mathbf{B}\text{-Loc}\, F$, there exists $f' \in F^{(n)}$ such that

$$\begin{aligned} f \circ e &= f \circ \langle \pi_1^n, \dots, \pi_n^n \rangle \circ e \\ &= f \circ \langle \pi_1^n \circ e, \dots, \pi_n^n \circ e \rangle \\ &= f' \circ \langle \pi_1^n \circ e, \dots, \pi_n^n \circ e \rangle \\ &= f' \circ \langle \pi_1^n, \dots, \pi_n^n \rangle \circ e \\ &= f' \circ e. \end{aligned}$$

Since e is an epimorphism, it follows $f = f'$, which establishes $(\mathrm{Loc}^{\mathbb{T}} F)^{(n)} = F^{(n)}$. $\square$

Evidently, this implies $\mathrm{Loc}^{\mathbb{T}} F = F$ for all $F \subseteq O_{\mathbf{A}}$ if, for each $n \in \mathbb{N}_+$, there exists an epimorphism from some $\mathbf{B} \in \mathbb{T}$ to $\mathbf{A}^n$. Furthermore, if we are only interested in the local closures of the clones, a weaker condition is sufficient.

Corollary 3.21. *If one of the following two conditions hold, then we have $\mathrm{Loc}^{\mathbb{T}} C = C$ for all $C \leq O_{\mathbf{A}}$:*

(i) For each $k \in \mathbb{N}_+$, there exists $n \geq k$ such that $\mathbf{A}^n \leqslant \mathbf{B}$ for some $\mathbf{B} \in \mathbb{T}$.
(ii) Each $f \in O_{\mathbf{A}}$ is essentially at most n-ary and $\mathbf{A}^n \leqslant \mathbf{B}$ for some $\mathbf{B} \in \mathbb{T}$.

Proof. (i) We only need to show $\mathrm{Loc}^{\mathbb{T}} C \subseteq C$. Let $k \in \mathbb{N}_+$ and let $f \in (\mathrm{Loc}^{\mathbb{T}} C)^{(k)}$. By assumption, there exists $n \geq k$ such that $\mathbf{A}^n \leqslant \mathbf{B}$ for some $\mathbf{B} \in \mathbb{T}$. Let f' be the n-ary operation that arises from f by adding $n - k$ nonessential arguments. Clearly, $f' \in \mathrm{Loc}^{\mathbb{T}} C$ and we can apply the last lemma to obtain $f' \in C$, which implies $f \in C$.

(ii) As we have remarked above, C and $\mathrm{Loc}^{\mathbb{T}} C$ are both clones over $\mathbf{A}$. By Lemma 3.20, they coincide on their n-ary parts. Since each operation among $O_{\mathbf{A}}$ is essentially at most n-ary, this means $C = \mathrm{Loc}^{\mathbb{T}} C$. $\square$

For relations, the following statement is obvious:

Proposition 3.22. *Let $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$. If all relations in R are finite, then $\mathrm{LOC}^{\mathbb{T}} R = R$.*

Thus, we have $\mathrm{LOC}^{\mathbb{T}} R = R$ for all $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ if there are only finitely many morphisms from $\mathbf{B}$ to $\mathbf{A}$ for each $\mathbf{B} \in \mathbb{T}$. The following lemma shows that this is also a necessary condition:

Lemma 3.23. *We have $\mathrm{LOC}^{\mathbb{T}} R = R$ for all $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ if and only if $\mathcal{C}(\mathbf{B}, \mathbf{A})$ is finite for all $\mathbf{B} \in \mathbb{T}$.*

Proof. We only need to show "$\Longrightarrow$" since "$\Longleftarrow$" follows directly from Proposition 3.22. Let $\mathbf{B} \in \mathbb{T}$ such that $|\mathcal{C}(\mathbf{B}, \mathbf{A})| = \infty$. Define

$$R := \bigcup_{s \in \mathbb{N}_+} \{\sigma \in \mathrm{R}_{\mathbf{A}}^{(\mathbf{B})} \mid |\sigma| \leq s\}.$$

Now, let σ be the full relation $\mathcal{C}(\mathbf{B}, \mathbf{A})$ (or any other infinite relation of type $\mathbf{B}$). Clearly, we have $\sigma \notin R$. However, for each $s \in \mathbb{N}_+$ and $B \subseteq \sigma$ with $|B| \leq s$, we have $B \in R$. Hence, for $\sigma' := B$, we obtain $B \subseteq \sigma' \subseteq \sigma$. Thus, $\sigma \in \mathrm{LOC}^{\mathbb{T}} R$. $\square$

The following examples show that Lemma 3.20 and Lemma 3.23 generalize an observation for the local closure operators in the universal algebra case:

Examples 3.24.

(i) If $\mathcal{C} = \mathcal{S}et$ and $\mathbb{T} := \{\{1, \ldots, k\} \mid k \in \mathbb{N}_+\}$, then Lemma 3.23 establishes that we have $\mathrm{LOC}^{\mathbb{T}} R = R$ for all $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ if and only if $\mathbf{A}$ is a finite set. Furthermore, Lemma 3.20 yields that $\mathbf{A}$ being a finite set implies $\mathrm{Loc}^{\mathbb{T}} F = F$ for all $F \subseteq O_{\mathbf{A}}$. An easy proof shows that the other direction is also true. Thus, both local closure operators can be dismissed if and only if $\mathbf{A}$ is a finite set.

(ii) If $(\mathcal{C}, U)$ is a concrete category and $\mathbb{T}$ is a representation system of

$$\{\mathbf{A} \in \mathcal{C} \mid |U(\mathbf{A})| < \infty\}/\cong,$$

then Lemma 3.23 establishes that we have $\mathrm{LOC}^{\mathbb{T}} R = R$ for all $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ if $U(\mathbf{A})$ is a finite set. Furthermore, Lemma 3.20 yields $\mathrm{Loc}^{\mathbb{T}} F = F$ for all $F \subseteq O_{\mathbf{A}}$ if we assume that $U(\mathbf{A}^n)$ is a finite set for all $n \in \mathbb{N}_+$.

(iii) If $\mathbb{T}$ is a skeleton of $\mathcal{C}$, then Lemma 3.20 establishes $\mathrm{Loc}^{\mathbb{T}} F = F$ for all $\mathbf{A} \in \mathcal{C}$ and $F \subseteq O_{\mathbf{A}}$.

(iv) If $\mathcal{C}$ is the category of finite distributive lattices and we define the typeclass by setting $\mathbb{T} := \{\langle \mathfrak{P}(\{1, \ldots, k\}), \cup, \cap \rangle \mid k \in \mathbb{N}_+\}$, then Lemma 3.23 yields $\mathrm{LOC}^{\mathbb{T}} R = R$ for all $\mathbf{A} \in \mathcal{C}$ and $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$. Moreover, Lemma 3.20 establishes $\mathrm{Loc}^{\mathbb{T}} F = F$ for all $F \subseteq O_{\mathbf{A}}$ whenever $\mathbf{A}$ is a Boolean lattice. It is possible (but not very easy) to give a direct proof that $\mathbf{A}$ being a Boolean lattice is, in fact, equivalent to having $\mathrm{Loc}^{\mathbb{T}} F = F$ for all $F \subseteq O_{\mathbf{A}}$. However, we will see in Example 4.15 (page 24) that this is one of the statements that are much easier to solve after dualizing them.

Case (ii) implies that we can always choose $\mathbb{T}$ such that we have $\mathrm{Loc}^{\mathbb{T}} F = F$ for all $F \subseteq O_{\mathbf{A}}$ (i.e., $\mathrm{Loc}^{\mathbb{T}}$ becomes obsolete). However, we cannot necessarily choose $\mathbb{T}$ such that we have $\mathrm{LOC}^{\mathbb{T}} R = R$ for all $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$. This somewhat unsymmetrical behaviour could be avoided by allowing operations of infinite arity, that is, we had to define $O_{\mathbf{A}}$ to be the class of morphisms from any non-empty power of $\mathbf{A}$ to $\mathbf{A}$.

Now, we will show that many lemmas that hold for Pol-Inv hold almost verbatim for $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$.

Lemma 3.25. *Let $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ and $F \subseteq O_{\mathbf{A}}$. Then, $\mathrm{Pol}_{\mathbf{A}} R$ and $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F$ are a clone of operations and a clone of relations, respectively. That is, we have*

(i) $\mathrm{Clo}\,(\mathrm{Pol}_{\mathbf{A}} R) = \mathrm{Pol}_{\mathbf{A}} R$,
(ii) $\mathrm{Clo}^{\mathbb{T}}\left(\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F\right) = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F$.

Proof. (i) Let $\sigma \in R$. Since the projection morphisms preserve σ and the superposition of operations preserving σ also preserves σ (see Proposition 3.4, page 8), $\mathrm{Pol}_{\mathbf{A}} R$ is a clone.

(ii) It is obvious that the empty relation $\emptyset$ is preserved by each $f \in F$. It remains to show that, for $f \in F$, the general superposition of relations preserved by f is again preserved by f. To this end, let I be an index class, let $\sigma_i \in R^{(\mathbf{B}_i)}$ $(i \in I)$ and let $\varphi\colon \mathbf{B} \to \mathbf{C}$ and $\varphi_i\colon \mathbf{B}_i \to \mathbf{C}$ be morphisms where $\mathbf{C} \in \mathcal{C}$, $\mathbf{B} \in \mathbb{T}$. Assume $s_1, \ldots, s_n \in \bigwedge_{(\varphi_i)}^{\varphi}(\sigma_i)$. Then, for each $j \in \{1, \ldots, n\}$, there exists $r_j \in \mathcal{C}(\mathbf{B}, \mathbf{A})$ such that $s_j = r_j \circ \varphi$ and $r_j \circ \varphi_i \in \sigma_i$ for all $i \in I$. Since f preserves each σ_i, we also have

$$f \circ \langle r_1, \ldots, r_n \rangle \circ \varphi_i = f \circ \langle \underbrace{r_1 \circ \varphi_i}_{\in \sigma_i}, \ldots, \underbrace{r_n \circ \varphi_i}_{\in \sigma_i} \rangle \in \sigma_i$$

for all $i \in I$. Thus, we have $f \circ \langle r_1, \ldots, r_n \rangle \circ \varphi \in \bigwedge_{(\varphi_i)}^{\varphi}(\sigma_i)$, whence it follows

$$f \circ \langle s_1, \ldots, s_n \rangle = f \circ \langle r_1 \circ \varphi, \ldots, r_n \circ \varphi \rangle = f \circ \langle r_1, \ldots, r_n \rangle \circ \varphi \in \bigwedge_{(\varphi_i)}^{\varphi}(\sigma_i).$$

Note that this proof is also valid for the case $I = \emptyset$. ☐

Lemma 3.26. *Let $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, $F \subseteq O_{\mathbf{A}}$, $n, s \in \mathbb{N}_+$ and let $\mathbf{C} \in \mathbb{T}$. Then, the following statements hold for all $1 \leq n \leq s$ and $\mathbf{B} \in \mathbb{T}$ where $\mathbf{B} \leqslant \mathbf{C}$:*

(i) $\mathrm{Pol}_{\mathbf{A}}^{(n)} R = \mathrm{Pol}_{\mathbf{A}}^{(n)} \mathrm{Clo}^{\mathbb{T}}(R) = \mathrm{Pol}_{\mathbf{A}}^{(n)} \mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R) = \mathrm{Pol}_{\mathbf{A}}^{(n)}\, \text{s-}\mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$,
(ii) $\mathrm{Pol}_{\mathbf{A}} R = \mathrm{Pol}_{\mathbf{A}} \mathrm{Clo}^{\mathbb{T}}(R) = \mathrm{Pol}_{\mathbf{A}} \mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$,
(iii) $\mathrm{Inv}_{\mathbf{A}}^{(\mathbf{B})} F = \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{B})} \mathrm{Clo}(F) = \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{B})} \mathrm{Loc}^{\mathbb{T}} \mathrm{Clo}(F) = \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{B})}\, \mathbf{C}\text{-}\mathrm{Loc}\, \mathrm{Clo}(F)$,
(iv) $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Clo}(F) = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Loc}^{\mathbb{T}} \mathrm{Clo}(F)$.

Proof. (i) It is easy to see that the sets in (i) form a decreasing chain from the left to the right. For the other direction, let $f \in \mathrm{Pol}_{\mathbf{A}}^{(n)} R$. We have to show that f belongs to $\mathrm{Pol}_{\mathbf{A}}^{(n)}\, \text{s-}\mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$, i.e., f preserves each $\sigma \in \text{s-}\mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$. Since $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} R$ is a clone of relations by Lemma 3.25 (ii), we have

$$\mathrm{Clo}^{\mathbb{T}}(R) \subseteq \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} R,$$

and hence

$$\mathrm{Pol}_{\mathbf{A}} \mathrm{Clo}^{\mathbb{T}}(R) \supseteq \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} R = \mathrm{Pol}_{\mathbf{A}} R.$$

Thus, $f \in \mathrm{Pol}^{(n)}_{\mathbf{A}} \mathrm{Clo}^{\mathbb{T}}(R)$. Now let $\sigma \in$ s-$\mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$ be a relation of type $\mathbf{B}$ and let $r_1, \dots, r_n \in \sigma$. Since $n \leq s$, there exists some $\sigma' \in \mathrm{Clo}^{\mathbb{T}}(R)$ such that $\{r_1, \dots, r_n\} \subseteq \sigma' \subseteq \sigma$. Hence, $f \circ \langle r_1, \dots, r_n \rangle \in \sigma' \subseteq \sigma$, and we are done.

(ii) By (i), we have

$$\bigcup_{n \in \mathbb{N}_+} \mathrm{Pol}^{(n)}_{\mathbf{A}} R = \bigcup_{n \in \mathbb{N}_+} \mathrm{Pol}^{(n)}_{\mathbf{A}} \mathrm{Clo}^{\mathbb{T}}(R) = \bigcup_{n \in \mathbb{N}_+} \mathrm{Pol}^{(n)}_{\mathbf{A}} \mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$$

and, thus, $\mathrm{Pol}_{\mathbf{A}} R = \mathrm{Pol}_{\mathbf{A}} \mathrm{Clo}^{\mathbb{T}}(R) = \mathrm{Pol}_{\mathbf{A}} \mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$.

(iii) Again, it is easy to see that the sets in (iii) form a decreasing chain from the left to the right. It remains to show that we have $\sigma \in \mathrm{Inv}^{(\mathbf{B})}_{\mathbf{A}}\, \mathbf{C}\text{-}\mathrm{Loc}\,\mathrm{Clo}(F)$ whenever $\sigma \in \mathrm{Inv}^{(\mathbf{B})}_{\mathbf{A}} F$. We get $\sigma \in \mathrm{Inv}^{(\mathbf{B})}_{\mathbf{A}} \mathrm{Clo}(F)$ in the same way we got $f \in \mathrm{Pol}^{(n)}_{\mathbf{A}} \mathrm{Clo}^{\mathbb{T}}(R)$ in part (i). Now let $f \in \mathbf{C}\text{-}\mathrm{Loc}\,\mathrm{Clo}(F)$ be n-ary. By Proposition 3.19, we also have $f \in \mathbf{B}\text{-}\mathrm{Loc}\,\mathrm{Clo}(F)$. Let $r_1, \dots, r_n \in \sigma$. We find some $f' \in \mathrm{Clo}(F)$ such that

$$f \circ \langle r_1, \dots, r_n \rangle = f' \circ \langle r_1, \dots, r_n \rangle.$$

Since $f' \circ \langle r_1, \dots, r_n \rangle \in \sigma$, it follows $f \rhd \sigma$, and thus, $\sigma \in \mathrm{Inv}^{(\mathbf{B})}_{\mathbf{A}}\, \mathbf{C}\text{-}\mathrm{Loc}\,\mathrm{Clo}(F)$.

(iv) follows from (iii) in the same way that (ii) follows from (i). □

Among other results that we will see later, this lemma allows us to give a direct calculation of $\Gamma_F(\sigma)$.

Proposition 3.27. *Let $F \subseteq O_{\mathbf{A}}$ and $\sigma \in \mathrm{R}^{\mathbb{T}}_{\mathbf{A}}$. Then,*

$$\Gamma_F(\sigma) = \{f \circ \langle r_1, \dots, r_n \rangle \mid f \in \mathrm{Clo}(F)^{(n)}, n \in \mathbb{N}_+, r_1, \dots, r_n \in \sigma\}.$$

Proof. Let us denote the right-hand side by γ. First, we will prove $\Gamma_F(\sigma) \subseteq \gamma$ by showing $\gamma \in \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} F$ and $\sigma \subseteq \gamma$. In order to show $\gamma \in \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} F$, let $f \in F^{(n)}$ and $r_1, \dots, r_n \in \gamma$. Then, for all $i \in \{1, \dots, n\}$, there exists an operation $f_i \in \mathrm{Clo}(F)^{(k_i)}$ and $r_{i,1}, \dots, r_{i,k_i} \in \sigma$ such that $r_i = f_i \circ \langle r_{i,1}, \dots, r_{i,k_i} \rangle$. But now, we have

$$\begin{aligned} f \circ \langle r_1, \dots, r_n \rangle &= f \circ \langle f_1 \circ \langle r_{1,1}, \dots, r_{1,k_1} \rangle, \dots, f_n \circ \langle r_{n,1}, \dots, r_{n,k_n} \rangle \rangle \\ &= f \circ \langle\langle f_1, \dots, f_n \rangle\rangle \circ \langle r_{1,1}, \dots, r_{n,k_n} \rangle. \end{aligned}$$

Since clones are closed under expanded superposition (see Proposition 2.8, page 5), it follows that we have $f \circ \langle\langle f_1, \dots, f_n \rangle\rangle \in \mathrm{Clo}(F)$, and hence $f \circ \langle r_1, \dots, r_n \rangle \in \gamma$. Thus, $\gamma \in \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} F$. Moreover, we have $\sigma \subseteq \gamma$ since $\mathrm{Clo}(F)$ contains the projection morphisms. Thus, $\Gamma_F(\sigma) \subseteq \gamma$.
Conversely, we have $\Gamma_F(\sigma) \in \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} F$ by definition, and $\mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} F = \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} \mathrm{Clo}(F)$ by Lemma 3.26 (iv). Thus, $\Gamma_F(\sigma) \in \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} \mathrm{Clo}(F)$, which implies $\gamma \subseteq \Gamma_F(\sigma)$. □

As an obvious consequence, we have the following proposition:

Proposition 3.28. *For $\sigma = \{r_1, \dots, r_n\}$ and a clone $C \leq O_{\mathbf{A}}$, we have*

$$\Gamma_C(\sigma) = \{f \circ \langle r_1, \dots, r_n \rangle \mid f \in C^{(n)}\}.$$

Before we start to prove our main result, we need to introduce the notion of directedness.

Definition 3.29. For $s \geq 1$, a family $\mathcal{F}$ of sets is said to be *s-directed* if, for any $X_1, \ldots, X_s \in \mathcal{F}$ and $r_1 \in X_1, \ldots, r_s \in X_s$, there exists $Z \in \mathcal{F}$ such that $\{r_1, \ldots, r_s\} \subseteq Z$.

Lemma 3.30. *Let $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, let $\emptyset \neq \mathcal{F} \subseteq R^{(\mathbf{B})}$, and let $\mathcal{F}$ be s-directed for some $s \geq 1$. Then, $\bigcup \mathcal{F} \in \text{s-LOC}^{\mathbb{T}} R$.*

Proof. We have to show that, for all $B \subseteq \bigcup \mathcal{F}$ with $|B| \leq s$, there exists $\sigma' \in R$ such that $B \subseteq \sigma' \subseteq \bigcup \mathcal{F}$. For $B = \{b_1, \ldots, b_s\} \subseteq \bigcup \mathcal{F}$, there exist $X_1, \ldots, X_s \in \mathcal{F}$ such that $b_1 \in X_1, \ldots, b_s \in X_s$. Since $\mathcal{F}$ is s-directed, this implies that there exists $Z \in \mathcal{F} \subseteq R^{(\mathbf{B})}$ such that $\{b_1, \ldots, b_s\} \subseteq Z \subseteq \bigcup \mathcal{F}$. Thus, the claim follows for $\sigma' := Z$. $\square$

We have prepared everything to state the main results of this section - the characterization of the Galois closed subclasses of $O_{\mathbf{A}}$ and $\mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$.

Theorem 3.31 (Galois closed sets of operations over $\mathbf{A}$). *Let $F \subseteq O_{\mathbf{A}}$. Then,*

(i) $\mathrm{Loc}^{\mathbb{T}} \mathrm{Clo}(F) = \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F$,
(ii) $\mathbf{C}\text{-Loc}\,\mathrm{Clo}(F) = \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{C})} F$ *for every* $\mathbf{C} \in \mathbb{T}$.

Proof. (ii) Since $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$ is a Galois connection, we have

$$\mathbf{C}\text{-Loc}\,\mathrm{Clo}(F) \subseteq \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathbf{C}\text{-Loc}\,\mathrm{Clo}(F) \subseteq \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{C})} \mathbf{C}\text{-Loc}\,\mathrm{Clo}(F)$$

and, by Lemma 3.26 (iii), we also have

$$\mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{C})} \mathbf{C}\text{-Loc}\,\mathrm{Clo}(F) = \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{C})} F.$$

For the other direction, let $f \in \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{C})} F$ be an n-ary operation. In order to show $f \in \mathbf{C}\text{-Loc}\,\mathrm{Clo}(F)$, let $r_1, \ldots, r_n \in \mathcal{C}(\mathbf{C}, \mathbf{A})$ and set $\sigma := \{r_1, \ldots, r_n\}$. We have

$$f \circ \langle r_1, \ldots, r_n \rangle \in \Gamma_F(\sigma).$$

But now, by Proposition 3.28, we find some $f' \in \mathrm{Clo}(F)^{(n)}$ such that

$$f \circ \langle r_1, \ldots, r_n \rangle = f' \circ \langle r_1, \ldots, r_n \rangle,$$

which proves $f \in \mathbf{C}\text{-Loc}\,\mathrm{Clo}(F)$.

(i) By (ii), we have

$$\mathrm{Loc}^{\mathbb{T}} \mathrm{Clo}(F) = \bigcap_{\mathbf{C} \in \mathbb{T}} \mathbf{C}\text{-Loc}\,\mathrm{Clo}(F) = \bigcap_{\mathbf{C} \in \mathbb{T}} \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathbf{C})} F = \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F. \qquad \square$$

The following lemma will help us to prove a similar characterization for the Galois closed classes of relations:

Lemma 3.32. *Let $\mathbf{B} \in \mathbb{T}$, let $s \geq 1$ and let $R \subseteq \mathrm{R}^{\mathbb{T}}_{\mathbf{A}}$. For $F := \mathrm{Pol}_{\mathbf{A}} R$ and $S \subseteq \mathcal{C}(\mathbf{B}, \mathbf{A})$, $|S| \leq s$, we have $\Gamma_{F^{(s)}}(S) \in \mathrm{Clo}^{\mathbb{T}}(R)$.*

Proof. For $S = \emptyset$, we have $\Gamma_F(S) = \emptyset \in \mathrm{Clo}^{\mathbb{T}}(R)$, and we are done. Let $S = \{r_1, \ldots, r_s\}$ (note that $r_1, \ldots, r_s$ do not have to be pairwise distinct). We define

$$I := \{(r'_1, \ldots, r'_s, \sigma) \mid \sigma \in R, r'_1, \ldots, r'_s \in \sigma\}.$$

For each $i = (r'_1, \ldots, r'_s, \sigma) \in I$, set $\sigma_i := \sigma$, let $\mathbf{B}_i$ be the type of σ_i and define $\varphi_i \colon \mathbf{B}_i \to \mathbf{A}^s$ by setting $\varphi_i := \langle r'_1, \ldots, r'_s \rangle$. Moreover, define $\varphi \colon \mathbf{B} \to \mathbf{A}^s$ by setting $\varphi := \langle r_1, \ldots, r_s \rangle$.

We shall prove that

$$\varrho_S := \bigwedge\nolimits^{\varphi}_{(\varphi_i)_{i\in I}} (\sigma_i) = \{r \circ \varphi \mid \forall\, i \in I : r \circ \varphi_i \in \sigma_i,\ r \in \mathcal{C}(\mathbf{A}^s, \mathbf{A})\} = \Gamma_{F^{(s)}}(S),$$

which would finish the proof since $\varrho_S \in \mathrm{Clo}^{\mathbb{T}}(R)$.

"$\subseteq$". Let $\varkappa \in \varrho_S$. We will start by showing that there exists $f \in F^{(s)} = \mathrm{Pol}^{(s)}_{\mathbf{A}} R$ such that $\varkappa = f \circ \varphi$: Since $\varkappa \in \varrho_S$, there exists $r \in \mathcal{C}(\mathbf{A}^s, \mathbf{A})$ such that $\varkappa = r \circ \varphi$ and $r \circ \varphi_i \in \sigma_i$ for all $i \in I$. Thus, for $f := r$, we obtain

$$\varkappa = r \circ \varphi = f \circ \varphi.$$

It remains to show that we have $f \in F = \mathrm{Pol}_{\mathbf{A}} R$, i.e., f preserves each $\sigma \in R$. Let $\sigma \in R$ and let $r'_1, \ldots, r'_s \in \sigma$. For $i := (r'_1, \ldots, r'_s, \sigma)$ we have $i \in I$, and hence

$$f \circ \langle r'_1, \ldots, r'_s \rangle = r \circ \langle r'_1, \ldots, r'_s \rangle = r \circ \varphi_i \in \sigma.$$

Thus, $f \in \mathrm{Pol}_{\mathbf{A}} R = F$, as required. By Proposition 3.27, it follows

$$\varkappa = f \circ \varphi = f \circ \langle r_1, \ldots, r_s \rangle \in \Gamma_{F^{(s)}}(S).$$

"$\supseteq$". Let $\widehat{r} \in \Gamma_{F^{(s)}}(S)$. Recall that, by Lemma 3.25, $F = \mathrm{Pol}_{\mathbf{A}} R$ is a clone of operations. Hence, $\mathrm{Clo}(F^{(s)}) \subseteq F$. It follows by Proposition 3.27 that $\widehat{r}$ is of the form $f \circ \langle r_{i_1}, \ldots, r_{i_l} \rangle$ for some $f \in F^{(l)}$ and $i_1, \ldots, i_l \in \{1, \ldots, s\}$. Then,

$$\widehat{r} = f \circ \langle r_{i_1}, \ldots, r_{i_l} \rangle = f \circ \langle \pi^s_{i_1}, \ldots, \pi^s_{i_l} \rangle \circ \langle r_1, \ldots, r_s \rangle.$$

Now, let $r := f \circ \langle \pi^s_{i_1}, \ldots, \pi^s_{i_l} \rangle$. Then,

$$\widehat{r} = r \circ \langle r_1, \ldots, r_s \rangle = r \circ \varphi,$$

and we can finish the proof by showing that we have $r \circ \varphi \in \varrho_S$. Indeed, for each $i = (r'_1, \ldots, r'_s, \sigma_i) \in I$, we get

$$r \circ \varphi_i = r \circ \langle r'_1, \ldots, r'_s \rangle \in \sigma_i$$

since $r \in \mathrm{Pol}_{\mathbf{A}} R$ and $r'_1, \ldots, r'_s \in \sigma_i \in R$. This implies $r \circ \varphi \in \varrho_S$, and we have established the desired result. $\square$

Theorem 3.33 (Galois closed classes of relations)**.** *Let* $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$*. Then,*

(i) $\mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R) = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} R$*,*
(ii) $\text{s-LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R) = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}}^{(s)} R$ *for every* $s \geq 1$*.*

Proof. (ii) Since $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$ is a Galois connection, we have

$$\text{s-LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R) \subseteq \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} \text{s-LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R) \subseteq \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}}^{(s)} \text{s-LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$$

and, by Lemma 3.26 (i), we also have

$$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}}^{(s)} \text{s-LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R) = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}}^{(s)} R.$$

For the other direction, let $\sigma \in \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}}^{(s)} R$ be a relation of type $\mathbf{B}$. We have to show $\sigma \in \text{s-LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$. By Proposition 3.27, we have $\sigma = \bigcup \mathcal{F}$ where

$$\mathcal{F} := \{\Gamma_{F^{(s)}}(B) \mid B \subseteq \sigma, |B| \leq s\}.$$

Clearly, $\mathcal{F}$ is non-empty and s-directed. Let $F := \mathrm{Pol}_{\mathbf{A}} R$. By Lemma 3.32, we get $\Gamma_{F^{(s)}}(B) \in \mathrm{Clo}^{\mathbb{T}}(R)$ for each $B \subseteq \sigma$ with $|B| \leq s$. In other words, $\mathcal{F} \subseteq \mathrm{Clo}^{\mathbb{T}}(R)^{(\mathbf{B})}$. Applying Lemma 3.30, we get $\sigma = \bigcup \mathcal{F} \in \text{s-LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R)$.

(i) By (ii), we have

$$\mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R) = \bigcap_{s \in \mathbb{N}_+} \text{s-LOC}^{\mathbb{T}} \mathrm{Clo}^{\mathbb{T}}(R) = \bigcap_{s \in \mathbb{N}_+} \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}}^{(s)} R = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} R. \qquad \square$$

The last two theorems enable us to characterize those subsets $F \subseteq O_{\mathbf{A}}$ and those subclasses $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ which can be represented as $\mathrm{Pol}_{\mathbf{A}} R'$ and $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F'$ for some $R' \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ and $F' \subseteq O_{\mathbf{A}}$, respectively.

Corollary 3.34. *For* $F \subseteq O_{\mathbf{A}}$*, the following are equivalent:*

(1) $F \leq O_{\mathbf{A}}$ *(i.e.,* $F = \mathrm{Clo}(F)$*) and* $\mathrm{Loc}^{\mathbb{T}} F = F$*.*
(2) $F = \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F$*.*
(3) $\exists\, R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}} : F = \mathrm{Pol}_{\mathbf{A}} R$*.*

Proof. (1) $\Longrightarrow$ (2) by Theorem 3.31.
(2) $\Longrightarrow$ (3) is trivial.
(3) $\Longrightarrow$ (1). We have $F = \mathrm{Pol}_{\mathbf{A}} R \leq O_{\mathbf{A}}$ by Lemma 3.25. By Theorem 3.31, we also have $\mathrm{Loc}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} R = \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} R = \mathrm{Pol}_{\mathbf{A}} R$. $\square$

Corollary 3.35. *For* $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$*, the following are equivalent:*

(1) $R \leq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ *(i.e.,* $R = \mathrm{Clo}^{\mathbb{T}}(R)$*) and* $\mathrm{LOC}^{\mathbb{T}} R = R$*.*
(2) $R = \mathrm{Inv}\, \mathrm{Pol}_{\mathbf{A}} R$*.*
(3) $\exists\, F \subseteq O_{\mathbf{A}} : R = \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F$*.*

Proof. (1) $\Longrightarrow$ (2) by Theorem 3.33.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1). We have $R = \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} F \leq \mathrm{R}^{\mathbb{T}}_{\mathbf{A}}$ by Lemma 3.25. By Theorem 3.33, we also have $\mathrm{LOC}^{\mathbb{T}} \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} F = \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} R = \mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}} R$. $\square$

By Corollary 3.21 and Lemma 3.23 (page 13), we can also state the following corollary:

Corollary 3.36. *Assume that the set of morphisms from any* $\mathbf{B} \in \mathbb{T}$ *to* $\mathbf{A}$ *is finite and that, for each* $k \in \mathbb{N}_+$, *there exists* $n \geq k$ *such that* $\mathbf{A}^n \leqslant \mathbf{B}$ *for some* $\mathbf{B} \in \mathbb{T}$. *Then, the Galois closed subclasses of* $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}}$ *are precisely the clones of operations and the clones of relations, respectively. Consequently,* $\mathcal{L}_{\mathbf{A}}$ *and* $\mathcal{L}^{*\mathbb{T}}_{\mathbf{A}}$ *are dually isomorphic via* $\mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}}$.

Note that this corollary generalizes the fact that, in universal algebra, the lattice of clones and the lattice of clones of relations are dually isomorphic if they are defined on a finite set A.

If the conditions of the above corollary are not satisfied, then we have to adjust the result:

Definition 3.37. Denote by $\mathrm{Loc}^{\mathbb{T}} \mathcal{L}_{\mathbf{A}}$ the lattice of locally closed clones of operations over $\mathbf{A}$ and by $\mathrm{LOC}^{\mathbb{T}} \mathcal{L}^{*\mathbb{T}}_{\mathbf{A}}$ the lattice of locally closed clones of relations on $\mathbf{A}$.

Corollary 3.38. $\mathrm{Loc}^{\mathbb{T}} \mathcal{L}_{\mathbf{A}}$ *and* $\mathrm{LOC}^{\mathbb{T}} \mathcal{L}^{*\mathbb{T}}_{\mathbf{A}}$ *are dually isomorphic via* $\mathrm{Inv}^{\mathbb{T}}_{\mathbf{A}}$.

3.1.2 A Remark on the Choice of the Typeclass

As we have seen, the choice of $\mathbb{T}$ influences the local closure operators, the clones of relations, and consequently, the Galois closed classes of operations as well as those of relations.

In this section, we will discuss how to choose $\mathbb{T}$ such that the local closure operators share a certain behaviour with the local closure operators of the usual Pol-Inv. Recall that our framework coincides with the classical case if $\mathcal{C}$ is the category of sets and we choose $\mathbb{T} = \{\{1, \ldots, k\} \mid k \in \mathbb{N}_+\}$. For brevity, we write $\underline{k}$ instead of $\{1, \ldots, k\}$. Evidently, we have $\underline{k_1} \leqslant \underline{k_2}$ (i.e., there exists an epimorphism from $\underline{k_2}$ to $\underline{k_1}$) whenever $k_1 \leq k_2$. Thus, for $F \subseteq O_{\mathbf{A}}$, Proposition 3.19 (page 12) yields

$$\underline{1}\text{-}\mathrm{Loc}^{\mathbb{T}} F \supseteq \underline{2}\text{-}\mathrm{Loc}^{\mathbb{T}} F \supseteq \underline{3}\text{-}\mathrm{Loc}^{\mathbb{T}} F \supseteq \ldots$$

and

$$\mathrm{Loc}^{\mathbb{T}} F = \bigcap_{k \in \mathbb{N}_+} \underline{\mathrm{k}}\text{-}\mathrm{Loc}^{\mathbb{T}} F.$$

Roughly speaking, $\bigcap_{i=1}^{n} \underline{\mathrm{k}}\text{-}\mathrm{Loc}^{\mathbb{T}} F$ converges to $\mathrm{Loc}^{\mathbb{T}} F$ for $n \to \infty$. A similar statement can be formulated for $\mathrm{LOC}^{\mathbb{T}}$. We have

$$1\text{-}\mathrm{LOC}^{\mathbb{T}} R \supseteq 2\text{-}\mathrm{LOC}^{\mathbb{T}} R \supseteq 3\text{-}\mathrm{LOC}^{\mathbb{T}} R \supseteq \ldots$$

and

$$\mathrm{LOC}^{\mathbb{T}} R = \bigcap_{s \in \mathbb{N}_+} \text{s-}\mathrm{LOC}^{\mathbb{T}} R.$$

The statement about $\mathrm{LOC}^{\mathbb{T}}$ holds in any category, but we cannot necessarily order the objects in $\mathbb{T}$ such that we obtain a property analogue to the statement about $\mathrm{Loc}^{\mathbb{T}}$ from above. However, we can do so if $\mathbb{T}$ is a countable set of objects that is totally ordered by $\leqslant$ and has a minimum element.

Proposition 3.39. *Let $F \subseteq O_{\mathbf{A}}$ and let $\mathbb{T} = (\mathbf{C}_i)_{i \in \mathbb{N}_+} \subseteq \mathcal{C}$ with $\mathbf{C}_i \leqslant \mathbf{C}_j$ if and only if $i \leq j$. Then*

$$\mathbf{C}_1\text{-}\mathrm{Loc}\, F \supseteq \mathbf{C}_2\text{-}\mathrm{Loc}\, F \supseteq \mathbf{C}_3\text{-}\mathrm{Loc}\, F \supseteq \dots$$

and

$$\mathrm{Loc}^{\mathbb{T}} F = \bigcap_{i \in \mathbb{N}_+} \mathbf{C}_{\mathrm{i}}\text{-}\mathrm{Loc}\, F.$$

Proof. Note that we can have $\mathbf{C}_i \leqslant \mathbf{C}_j$ and $\mathbf{C}_j \leqslant \mathbf{C}_i$ if and only if $i = j$. Thus, $\mathbf{C}_j \cong \mathbf{C}_i$ can only occur for $i = j$, and $\mathbb{T}$ is a typeclass. The rest follows directly from Proposition 3.19 and the definition of $\mathrm{Loc}^{\mathbb{T}} F$. $\square$

4 A General Galois Theory for Dual Operations and Dual Relations

In this section, we will dualize the results from the last section to obtain a general Galois theory for dual operations and something that we will introduce as dual relations.

To this end, let $\mathcal{C}$ be a category that contains an object $\mathbf{X}$ and all finite non-empty copowers of $\mathbf{X}$. Recall that an n-ary dual operation over $\mathbf{X}$ is an n-ary operation over $\mathbf{X}$ in $\mathcal{C}^{op}$. Furthermore, $\mathbb{T}$ is a typeclass of $\mathcal{C}$ if and only if it is a typeclass of $\mathcal{C}^{op}$. Therefore, we can dualize all the definitions from the last section to obtain a Galois connection $\overline{\mathrm{Pol}}_{\mathbf{X}}\text{-}\overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}$ between sets of dual operations and classes of dualized relations. By the Duality Principle, it follows that the Galois closed classes are precisely the dualized local closures of clones of dual operations and the dualized local closures of the dualized clones of relations. This will be described in the upcoming two subsections.

4.1 Dual Relations

For the whole section, let $\mathbb{T} \subseteq \mathcal{C}$ be a typeclass.

Definition 4.1. Let $\mathbf{Y} \in \mathcal{C}$. A *dual relation of type* $\mathbf{Y}$ *on* $\mathbf{X}$ is a subset of $\mathcal{C}(\mathbf{X}, \mathbf{Y})$. Denote the class of all dual relations of type $\mathbf{Y}$ on $\mathbf{X}$ by $\overline{\mathrm{R}}_{\mathbf{X}}^{(\mathbf{Y})}$. Moreover,

$$\overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}} := \bigcup_{\mathbf{Y} \in \mathbb{T}} \overline{\mathrm{R}}_{\mathbf{X}}^{(\mathbf{Y})}$$

is called *the class of dual relations of the typeclass* $\mathbb{T}$ *on* $\mathbf{X}$. For a class of dual relations $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$, let $R^{(\mathbf{Y})} := R \cap \overline{\mathrm{R}}_{\mathbf{X}}^{(\mathbf{Y})}$.

It is easy to see that a dual relation of type $\mathbf{Y} \in \mathcal{C}$ is a relation of type $\mathbf{Y}$ in $\mathcal{C}^{op}$. In other words, the notion of a dual relation is the dualized notion of a relation.

Example 4.2. Every dual relation of type $\{1, \dots, k\}$ on a set $\mathbf{X} \in \mathcal{S}et$ is a k-ary corelation as introduced in [PR00]. These corelations have been further studied in the context of clone theory (e.g., [MP00, MR01]) as well as in the context of classical co-algebras (e.g., [Dol00, Maš01]).

We will now dualize the remaining notions of Subsection 3.1:

Definition 4.3. Let σ be a dual relation of type $\mathbf{Y}$ on $\mathbf{X}$, and let g be an n-ary dual operation over $\mathbf{X}$. We say that σ is *invariant* for g or that g *preserves* σ, written $g \mathrel{\overline{\rhd}} \sigma$, if $[r_1, \dots, r_n] \circ g \in \sigma$ whenever $r_1, \dots, r_n \in \sigma$. Furthermore, we say that a set of dual operations $G \subseteq \overline{O}_{\mathbf{X}}$ *preserves* σ, written $G \mathrel{\overline{\rhd}} \sigma$, if every $g \in G$ preserves σ.

It is easy to check that the injection morphisms preserve any dual relation on $\mathbf{X}$. This also follows by the Duality Principle from the fact that every projection morphism preserves each relation on $\mathbf{X} \in \mathcal{C}^{op}$.

Definition 4.4. A class $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$ is called a *clone of dual relations of the typeclass* $\mathbb{T}$ *on* $\mathbf{X}$, written $R \leq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$, if

(i) $\emptyset \in R$,
(ii) R is closed under *general superposition*, that is, the following holds: Let I be an index class, let $\sigma_i \in R^{(\mathbf{Y}_i)}$ $(i \in I)$ and let $\varphi\colon \mathbf{Z} \to \mathbf{Y}$ and $\varphi_i\colon \mathbf{Z} \to \mathbf{Y}_i$ be morphisms where $\mathbf{Z} \in \mathcal{C}$ and $\mathbf{Y} \in \mathbb{T}$. Then, the dual relation $\overline{\bigwedge}^{\varphi}_{(\varphi_i)_{i\in I}}(\sigma_i)_{i\in I} \in \overline{\mathrm{R}}_{\mathbf{X}}^{(\mathbf{Y})}$ defined by
$$\overline{\bigwedge}^{\varphi}_{(\varphi_i)_{i\in I}}(\sigma_i)_{i\in I} := \{\varphi \circ r \mid \forall\, i \in I : \varphi_i \circ r \in \sigma_i,\ r \in \mathcal{C}(\mathbf{X}, \mathbf{Z})\}$$
belongs to R.

Again, $\overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$ is a clone of dual relations and the intersection of clones of dual relations is a clone of dual relations. Hence, the following notion is well-defined:

Definition 4.5. For each $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$, denote by $\overline{\mathrm{Clo}}^{\mathbb{T}}(R)$ the least clone of dual relations that contains R. It is called the clone of dual relations *generated by* R.

It follows that clones of dual relations also form a complete lattice with respect to inclusion.

Definition 4.6. Denote by $\overline{L}_{\mathbf{X}}^{*\mathbb{T}}$ the class of clones of dual relations of the typeclass $\mathbb{T}$ on $\mathbf{X}$. Then, $\overline{\mathcal{L}}_{\mathbf{X}}^{*\mathbb{T}} := \langle \overline{L}_{\mathbf{X}}^{*\mathbb{T}}, \subseteq \rangle$ is called the *lattice of clones of dual relations of the typeclass* $\mathbb{T}$ *on* $\mathbf{X}$.

Example 4.7. If $\mathcal{C}$ is the category of sets and we choose $\mathbb{T} := \{\{1, \dots, k\} \mid k \in \mathbb{N}_+\}$, then the notion of clones of dual relations and that of clones of corelations as introduced in [PR00] coincide in $\mathcal{C}$.

4.2 The Galois Connection $\overline{\mathrm{Pol}}_{\mathbf{X}}$-$\overline{\mathrm{Inv}}^{\mathbb{T}}_{\mathbf{X}}$

Definition 4.8. We define the two operators $\overline{\mathrm{Inv}}^{\mathbb{T}}_{\mathbf{X}} \colon \mathfrak{P}(\overline{O}_{\mathbf{X}}) \to \mathfrak{P}(\overline{\mathrm{R}}^{\mathbb{T}}_{\mathbf{X}})$ and $\overline{\mathrm{Pol}}_{\mathbf{X}} \colon \mathfrak{P}(\overline{\mathrm{R}}^{\mathbb{T}}_{\mathbf{X}}) \to \mathfrak{P}(\overline{O}_{\mathbf{X}})$ as follows: For $G \subseteq \overline{O}_{\mathbf{X}}$ and $R \subseteq \overline{\mathrm{R}}^{\mathbb{T}}_{\mathbf{X}}$, set

$$\begin{aligned}\overline{\mathrm{Inv}}^{\mathbb{T}}_{\mathbf{X}}\, G &:= \{\sigma \in \overline{\mathrm{R}}^{\mathbb{T}}_{\mathbf{X}} \mid \forall\, g \in G : g \mathrel{\overline{\rhd}} \sigma\},\\ \overline{\mathrm{Pol}}_{\mathbf{X}}\, R &:= \{g \in \overline{O}_{\mathbf{X}} \mid \forall\, \sigma \in R : g \mathrel{\overline{\rhd}} \sigma\}.\end{aligned}$$

For $\mathbf{Y} \in \mathcal{C}$ and $n \in \mathbb{N}_+$, we use the following notation:

$$\begin{aligned}\overline{\mathrm{Inv}}^{(\mathbf{Y})}_{\mathbf{X}}\, G &:= \{\sigma \in \overline{\mathrm{R}}^{(\mathbf{Y})}_{\mathbf{X}} \mid \forall\, g \in G : g \mathrel{\overline{\rhd}} \sigma\},\\ \overline{\mathrm{Pol}}^{(n)}_{\mathbf{X}}\, R &:= \overline{\mathrm{Pol}}_{\mathbf{X}}\, R \cap \overline{O}^{(n)}_{\mathbf{X}}.\end{aligned}$$

It is easy to see that these notions are the duals of the corresponding notions from the last section. Note that, for $\mathcal{C} = \mathcal{S}et$ and $\mathbb{T} = \{\{1, \ldots, k\} \mid k \in \mathbb{N}_+\}$, it follows directly from the observations in Example 4.7 that the Galois Connection $\overline{\mathrm{Pol}}_{\mathbf{X}}$-$\overline{\mathrm{Inv}}^{\mathbb{T}}_{\mathbf{X}}$ coincides with the Galois connection cPol-cInv that is presented in [PR00].

Examples 4.9. The following examples are the duals of the examples presented in 3.16 (note that, by the Priestley duality [Pri72], the category of finite distributive lattices is dually equivalent to the category if finite bounded posets):

(i) Let $\sigma := \{id_{\mathbf{X}}\}$. Then, $\overline{\mathrm{Pol}}_{\mathbf{X}}\{\sigma\}$ is the set of all dual idempotent operations over $\mathbf{X}$. That is, $\overline{\mathrm{Pol}}_{\mathbf{X}}\{\sigma\} = \{g \in \overline{O}_{\mathbf{X}} \mid [id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}] \circ g = g\}$.
(ii) Let $C \leq \overline{O}_{\mathbf{X}}$. Note that $C^{(n)}$ is a dual relation of type $n \cdot \mathbf{X}$. Now, $\overline{\mathrm{Pol}}_{\mathbf{X}}\, C^{(n)}$ is the largest clone C' that agrees with C on its n-ary part.
(iii) If two dual operations g, g' are essentially the same, then $\overline{\mathrm{Inv}}^{\mathbb{T}}_{\mathbf{X}}\{g\} = \overline{\mathrm{Inv}}^{\mathbb{T}}_{\mathbf{X}}\{g'\}$.
(iv) Let $\mathcal{C}$ be the category of finite bounded partially ordered sets and let $\mathbf{Y} \in \mathcal{C}$. Moreover, let $\sigma \in \overline{\mathrm{R}}^{(\mathbf{Y})}_{\mathbf{X}}$ contain all morphisms $r \colon n \cdot \mathbf{X} \to \mathbf{Y}$ with $r(x) \notin \{0^{\mathbf{Y}}, 1^{\mathbf{Y}}\}$ for all $x \notin \{0^{n \cdot \mathbf{X}}, 1^{n \cdot \mathbf{X}}\}$. Then, $\overline{\mathrm{Pol}}_{\mathbf{X}}\{\sigma\}$ is the set of all $g \in \overline{O}_{\mathbf{X}}$ such that we have $g(x) \notin \{0^{n \cdot \mathbf{X}}, 1^{n \cdot \mathbf{X}}\}$ for all $x \in \mathbf{X} \setminus \{0^{\mathbf{X}}, 1^{\mathbf{X}}\}$.

The only thing left to dualize are the local closure operators.

Definition 4.10. Let $G \subseteq \overline{O}_{\mathbf{X}}$, $R \subseteq \overline{\mathrm{R}}^{\mathbb{T}}_{\mathbf{X}}$, $s \geq 1$ and let $\mathbf{Z} \in \mathcal{C}$. We define the following local closure operators:

$$\begin{aligned}\overline{\mathbf{Z}\text{-}\mathrm{Loc}}\, G :=& \{g \in \overline{O}^{(n)}_{\mathbf{X}} \mid n \geq 1, \forall\, r_1, \ldots, r_n \in \mathcal{C}(\mathbf{X}, \mathbf{Z}) :\\ &\qquad \exists\, g' \in G : [r_1, \ldots, r_n] \circ g = [r_1, \ldots, r_n] \circ g'\},\\ \overline{s\text{-}\mathrm{LOC}}^{\mathbb{T}}\, R :=& \{\sigma \in \overline{\mathrm{R}}^{\mathbb{T}}_{\mathbf{X}} \mid \forall B \subseteq \sigma, |B| \leq s : \exists\, \sigma' \in R : B \subseteq \sigma' \subseteq \sigma\}.\end{aligned}$$

Furthermore, let

$$\overline{\mathrm{Loc}}^{\mathbb{T}}\, G := \bigcap_{\mathbf{Z} \in \mathbb{T}} \overline{\mathbf{Z}\text{-}\mathrm{Loc}}\, G$$

and

$$\overline{\mathrm{LOC}}^{\mathbb{T}} R := \bigcap_{s\in\mathbb{N}_+} \overline{\text{s-LOC}}^{\mathbb{T}} R.$$

In other words, $\overline{\mathbf{Z}\text{-Loc}}\, G$ is the set of all dual operations $g \in \overline{O}_{\mathbf{X}}$ that cannot be distinguished from a dual operation $g' \in G$ if a cotupling of n morphism from $\mathbf{X}$ to some $\mathbf{Z}$ is applied after g and g'. Moreover, $\overline{\text{s-LOC}}^{\mathbb{T}} R$ is the class of all dual relations $\sigma \in \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$ such that, for every $B \subseteq \sigma$ with at most s elements, there exists a member σ' of R that agrees with σ on B and is contained in σ.

We have dualized every definition of the last section. Thus, each proposition, lemma and theorem from Section 3 holds in its dualized version. For instance, we have the following statements (recall that we write $\mathbf{Z}_1 \leqslant \mathbf{Z}_2$ if there exists a monomorphism from $\mathbf{Z}_1$ to $\mathbf{Z}_2$):

Proposition 4.11. *If $\mathbf{Z}_1 \leqslant \mathbf{Z}_2$, then $\overline{\mathbf{Z}_2\text{-Loc}}\, G \subseteq \overline{\mathbf{Z}_1\text{-Loc}}\, G$ for all $G \subseteq \overline{O}_{\mathbf{X}}$.*

Proposition 4.12. *Let $n \in \mathbb{N}_+$ and let $G \subseteq \overline{O}_{\mathbf{X}}$. If we have $n \cdot \mathbf{X} \leqslant \mathbf{Y}$ for some $\mathbf{Y} \in \mathbb{T}$, then $(\overline{\mathrm{Loc}}^{\mathbb{T}} G)^{(n)} = G^{(n)}$ (that is, $\overline{\mathrm{Loc}}^{\mathbb{T}} G$ and G agree on their n-ary part).*

Corollary 4.13. *If one of the following two conditions hold, then we have $\overline{\mathrm{Loc}}^{\mathbb{T}} C = C$ for all $C \leq \overline{O}_{\mathbf{X}}$:*

(i) For each $k \in \mathbb{N}_+$, there exists $n \geq k$ such that $n \cdot \mathbf{X} \leqslant \mathbf{Y}$ for some $\mathbf{Y} \in \mathbb{T}$.
(ii) Each $g \in \overline{O}_{\mathbf{X}}$ is essentially at most n-ary and $n \cdot \mathbf{X} \leqslant \mathbf{Y}$ for some $\mathbf{Y} \in \mathbb{T}$.

Proposition 4.14. *We have $\overline{\mathrm{LOC}}^{\mathbb{T}} R = R$ for all $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$ if and only if $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ is finite for all $\mathbf{Y} \in \mathbb{T}$.*

Examples 4.15. Except (i), the following examples are the duals of the examples presented in 3.24 (page 14). Without using duality, they also follow from Proposition 4.12 and 4.14.

(i) If $\mathcal{C} = \mathcal{Set}$ and $\mathbb{T} := \{\{1, \dots, k\} \mid k \in \mathbb{N}_+\}$, then both local closure operators can be dismissed if and only if $\mathbf{X}$ is a finite set.

(ii) If $(\mathcal{C}, U)$ is a concrete category and $\mathbb{T}$ is a representation system of

$$\{\mathbf{X} \in \mathcal{C} \mid |U(\mathbf{X})| < \infty\}/\cong,$$

then $\mathrm{LOC}^{\mathbb{T}}$ can be dismissed if $U(\mathbf{X})$ is a finite set and $\mathrm{Loc}^{\mathbb{T}}$ can be dismissed if $U(n \cdot \mathbf{X})$ is a finite set for all $n \in \mathbb{N}_+$.

(iii) If $\mathcal{C}$ is the category of finite bounded posets and we define the typeclass by setting

$$\mathbb{T} := \{\langle\{0, a_1, \dots, a_k, 1\}, 0, 1, \leq\rangle \mid k \in \mathbb{N}_+,\ a_1, \dots, a_k \text{ antichain w.r.t. } \leq\},$$

then $\overline{\mathrm{LOC}}^{\mathbb{T}} R = R$ for all $\mathbf{X} \in \mathcal{C}$ and $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$. Moreover, we have $\overline{\mathrm{Loc}}^{\mathbb{T}} G = G$ for all $G \subseteq \overline{O}_{\mathbf{X}}$ if $\mathbf{X}$ is isomorphic to one of the posets among $\mathbb{T}$. Let us show that $\mathbf{X}$ being isomorphic to some $\mathbf{Y} \in \mathbb{T}$ is also a necessary condition. Assuming that $\mathbf{X}$

is not isomorphic to some object among $\mathbb{T}$ implies that there exist $x_1, x_2 \in \mathbf{X}$ such that $0^{\mathbf{X}} \neq x_1 < x_2 \neq 1^{\mathbf{X}}$. Define $g \in \overline{O}_{\mathbf{X}}^{(1)}$ by setting

$$g(x) := \begin{cases} 0^{\mathbf{X}} & \text{if } x \leq x_1, \\ 1^{\mathbf{X}} & \text{if } x \geq x_2, \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that g is well-defined. We will show $g \in \overline{\mathrm{Loc}}^{\mathbb{T}}(\overline{O}_{\mathbf{X}}^{(1)} \setminus \{g\})$. Take any $\mathbf{Y} \in \mathbb{T}$ and let $r \in \mathcal{C}(\mathbf{X}, \mathbf{Y})$. If r maps x_1, x_2 to the same element $y \in \mathbf{Y}$, then we have $r \circ g = r \circ g'$ for $g' \in \overline{O}_{\mathbf{X}} \setminus \{g\}$ defined as follows:

$$g'(x) := \begin{cases} x_1 & \text{if } x_1 \leq x \leq x_2, \\ x & \text{otherwise.} \end{cases}$$

If $r(x_1) \neq r(x_2)$, then we have $r(x_1) = 0^{\mathbf{Y}}$ or $r(x_2) = 1^{\mathbf{Y}}$. In the first case, we obtain $r \circ g = r \circ g'$ for $g' \in \overline{O}_{\mathbf{X}} \setminus \{g\}$ defined as follows:

$$g'(x) := \begin{cases} 1^{\mathbf{X}} & \text{if } x \geq x_2, \\ x & \text{otherwise.} \end{cases}$$

An analogue trick works for the case $r(x_2) = 1^{\mathbf{Y}}$. Thus, $g \in \overline{\mathrm{Loc}}^{\mathbb{T}}(\overline{O}_{\mathbf{X}}^{(1)} \setminus \{g\})$. Therefore, $\overline{\mathrm{LOC}}^{\mathbb{T}}\, G = G$ for all $G \subseteq \overline{O}_{\mathbf{X}}$ if and only if $\mathbf{X} \cong \mathbf{Y}$ for some $\mathbf{Y} \in \mathbb{T}$. By the Duality Principle, this also proves (iii) from Examples 3.24 since the elements of $\mathbb{T}$ are (up to isomorphism) the duals of Boolean lattices.

By the Duality Principle, we also immediately obtain our main theorems:

Theorem 4.16 (Galois closed sets of dual operations)**.** *Let $G \subseteq \overline{O}_{\mathbf{X}}$. Then,*

(i) $\overline{\mathrm{Loc}}^{\mathbb{T}}\, \overline{\mathrm{Clo}}(G) = \overline{\mathrm{Pol}}_{\mathbf{X}}\, \overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}\, G$,
(ii) $\overline{\mathbf{Z}\text{-}\mathrm{Loc}}\, \overline{\mathrm{Clo}}(G) = \overline{\mathrm{Pol}}_{\mathbf{X}}\, \overline{\mathrm{Inv}}_{\mathbf{X}}^{(\mathbf{Z})}\, G$ *for every* $\mathbf{Z} \in \mathbb{T}$.

Theorem 4.17 (Galois closed classes of dual relations)**.** *Let $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$. Then,*

(i) $\overline{\mathrm{LOC}}^{\mathbb{T}}\, \overline{\mathrm{Clo}}^{\mathbb{T}}(R) = \overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}\, \overline{\mathrm{Pol}}_{\mathbf{X}}\, R$,
(ii) $\overline{\text{s-LOC}}^{\mathbb{T}}\, \overline{\mathrm{Clo}}^{\mathbb{T}}(R) = \overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}\, \overline{\mathrm{Pol}}_{\mathbf{X}}^{(s)}\, R$ *for every* $s \geq 1$.

Thus, we are able to characterize those subsets $G \subseteq \overline{O}_{\mathbf{X}}$ and those subclasses $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$ which can be represented as $\overline{\mathrm{Pol}}_{\mathbf{X}}\, R'$ and $\overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}\, G'$ for some $R' \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$ and $G' \subseteq \overline{O}_{\mathbf{X}}$, respectively.

Corollary 4.18. *For $G \subseteq \overline{O}_{\mathbf{X}}$, the following are equivalent:*

(1) $G \leq \overline{O}_{\mathbf{X}}$ *(i.e.,* $G = \overline{\mathrm{Clo}}(G)$*) and* $\overline{\mathrm{Loc}}^{\mathbb{T}}\, G = G$.
(2) $G = \overline{\mathrm{Pol}}_{\mathbf{X}}\, \overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}\, G$.

(3) $\exists\, R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}} : G = \overline{\mathrm{Pol}}_{\mathbf{X}}\, R$.

Corollary 4.19. *For $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$, the following are equivalent:*

(1) $R \leq \overline{\mathrm{R}}_{\mathbf{X}}^{\mathbb{T}}$ *(i.e., $R = \overline{\mathrm{Clo}}^{\mathbb{T}}(R)$) and* $\overline{\mathrm{LOC}}^{\mathbb{T}}\, R = R$.
(2) $R = \overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}\, \overline{\mathrm{Pol}}_{\mathbf{X}}\, R$.
(3) $\exists\, G \subseteq \overline{O}_{\mathbf{X}} : R = \overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}\, G$.

For the sake of completeness, let us also list the following, now obvious results:

Corollary 4.20. *Assume that the set of morphisms from $\mathbf{X}$ to any $\mathbf{Y} \in \mathbb{T}$ is finite and that, for each $k \in \mathbb{N}_+$, there exists $n \geq k$ such that $n \cdot \mathbf{X} \leqslant \mathbf{Y}$ for some $\mathbf{Y} \in \mathbb{T}$. Then, the Galois closed subclasses of $\overline{\mathrm{Pol}}_{\mathbf{X}}$-$\overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}$ are precisely the clones of dual operations and the clones of dual relations, respectively. Consequently, $\overline{\mathcal{L}}_{\mathbf{X}}$ and $\overline{\mathcal{L}}_{\mathbf{X}}^{*\mathbb{T}}$ are dually isomorphic via $\overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}$.*

This corollary generalizes the result from [PR00] which states that the lattice of clones of cofunctions on a finite set A and that of corelations on A are dually isomorphic.

Definition 4.21. Denote by $\overline{\mathrm{Loc}}^{\mathbb{T}}\, \overline{\mathcal{L}}_{\mathbf{X}}$ and $\overline{\mathrm{LOC}}^{\mathbb{T}}\, \overline{\mathcal{L}}_{\mathbf{X}}^{*\mathbb{T}}$ the lattice of locally closed clones of dual operations over $\mathbf{X}$ and the lattice of locally closed clones of dual relations on $\mathbf{X}$, respectively.

Corollary 4.22. $\overline{\mathrm{Loc}}^{\mathbb{T}}\, \overline{\mathcal{L}}_{\mathbf{X}}$ *and* $\overline{\mathrm{LOC}}^{\mathbb{T}}\, \overline{\mathcal{L}}_{\mathbf{X}}^{*\mathbb{T}}$ *are dually isomorphic via* $\overline{\mathrm{Inv}}_{\mathbf{X}}^{\mathbb{T}}$.

Concerning Subsection 3.1.2, we have analogue results. Of course, the objects in $\mathbb{T}$ do not have to form a chain with respect to $\leqslant$ but with respect to $\lessdot$.

Proposition 4.23. *Let $G \subseteq \overline{O}_{\mathbf{X}}$ and let $\mathbb{T} = (\mathbf{Z}_i)_{i \in \mathbb{N}_+} \subseteq \mathcal{C}$ with $\mathbf{Z}_i \lessdot \mathbf{Z}_j$ if and only if $i \leq j$. Then*

$$\overline{\mathbf{Z}_1\text{-Loc}}\, G \supseteq \overline{\mathbf{Z}_2\text{-Loc}}\, G \supseteq \overline{\mathbf{Z}_3\text{-Loc}}\, G \supseteq \dots$$

and

$$\overline{\mathrm{Loc}}^{\mathbb{T}}\, G = \bigcap_{i \in \mathbb{N}_+} \overline{\mathbf{Z}_{\mathrm{i}}\text{-Loc}}\, G.$$

5 Concluding remarks

We have developed a general Galois theory for operations and relations in arbitrary categories, and we have discussed its dualization. During this process, we have shown that our theory generalizes the classical case from universal algebra as well as other examples, such as the coalgebraic case from [PR00].

In our framework, we did not consider nullary operations. We made this decision because clone theory is usually pursued without constants, and so was the development of Pol-Inv to which our Galois connection was intended to be analogue. However, it should at least be remarked that our generalized theory can be modified by including

$\mathcal{C}(\mathbf{A}^0, \mathbf{A})$ into the definition of $O_{\mathbf{A}}$ (note that this requires $\mathcal{C}$ to contain a terminal object). We will not elaborate the exact consequences of this change, but it should be noted that the theory would stay essentially the same and the main results would hold almost verbatim. However, some minor adjustments would be necessary. For instance, the empty relation would not necessarily be preserved by a given set of operations. Hence, condition (i) had to be removed from the definition of a clone of relations, and the smallest relation (of any given type) preserved by a set of operations would not necessarily be the empty relation.

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